10. Odpowiedź skokowa elementu oscylacyjnego

$$G(s) = \frac{k\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{k\omega_n^2}{(s+\zeta\omega_n+j\omega_d)(s+\zeta\omega_n-j\omega_d)} = \\ = \frac{k\omega_n^2}{(s+\zeta\omega_n)^2+\omega_d^2} \tag{26}$$

przy czym $\omega_d = \omega_n\sqrt{1-\zeta^2}$ – tłumiona pulsacja naturalna

$$\begin{aligned} H(s) &= \frac{G(s)}{s} = \frac{k\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)s} = k\frac{s^2+2\zeta\omega_n s+\omega_n^2-s^2-2\zeta\omega_n s}{(s^2+2\zeta\omega_n s+\omega_n^2)s} = \\ &= k\left(\frac{1}{s}-\frac{s^2+2\zeta\omega_n s}{(s^2+2\zeta\omega_n s+\omega_n^2)s}\right) = k\left(\frac{1}{s}-\frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}\right) = \\ &= k\left(\frac{1}{s}-\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}-\frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}\right) = \\ &= k\left(\frac{1}{s}-\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}-\frac{\zeta\omega_n}{\omega_d}\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\right) \end{aligned} \tag{27}$$

$$\left| \begin{aligned} \mathcal{L}^{-1}\left\{\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}\right\} &= e^{-\zeta\omega_n t}\cos(\omega_d t)\mathbb{1}(t) \\ \mathcal{L}^{-1}\left\{\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\right\} &= e^{-\zeta\omega_n t}\sin(\omega_d t)\mathbb{1}(t) \end{aligned} \right| \tag{28}$$

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}\left\{H(s)\right\} = \mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\} = \\ &= k\left(1-e^{-\zeta\omega_n t}\cos(\omega_d t)-\frac{\zeta\omega_n}{\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t)\right)\mathbb{1}(t) = \\ &= k\left[1-e^{-\zeta\omega_n t}\left(\cos(\omega_d t)+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right)\right]\mathbb{1}(t) \end{aligned}$$

$$h(t) = k\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sqrt{1-\zeta^2}\cos(\omega_d t) + \zeta\sin(\omega_d t)\right)\right]\mathbb{1}(t) =$$

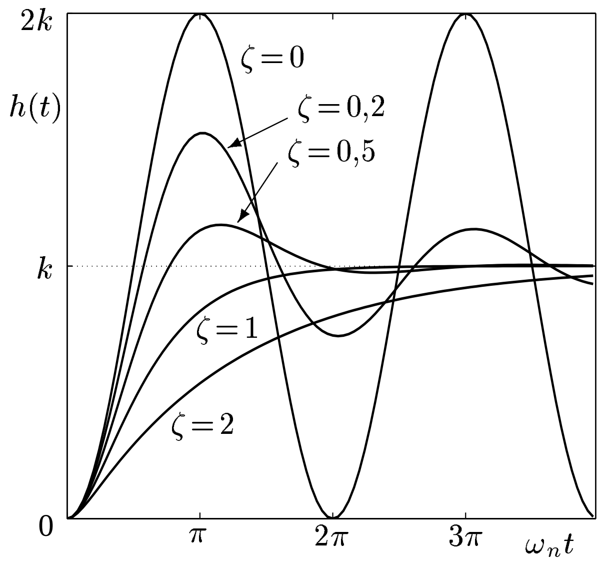
$$= \left|\begin{matrix}\zeta = \cos\theta \\ \sqrt{1-\zeta^2} = \sin\theta\end{matrix}\right| =$$

$$= k\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sin\theta\cos(\omega_d t) + \cos\theta\sin(\omega_d t)\right)\right]\mathbb{1}(t) =$$

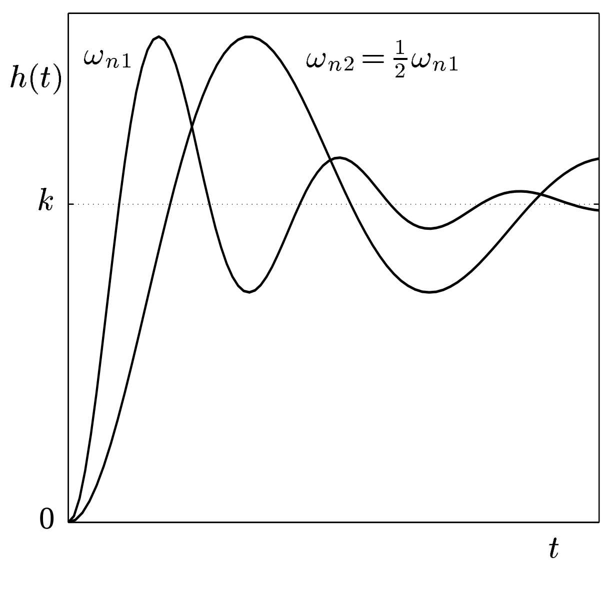
$$= k\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)\right]\mathbb{1}(t)$$

przy czym $\theta = \mathrm{arctg}\dfrac{\sin\theta}{\cos\theta} = \mathrm{arctg}\dfrac{\sqrt{1-\zeta^2}}{\zeta}$, $\quad \omega_d = \omega_n\sqrt{1-\zeta^2}$

$$h(t)\Big|_{\zeta=0} = k\left[1 - \sin\left(\omega_n t + \frac{\pi}{2}\right)\right]\mathbb{1}(t) = k\left[1 - \cos(\omega_n t)\right]\mathbb{1}(t) \quad (29)$$

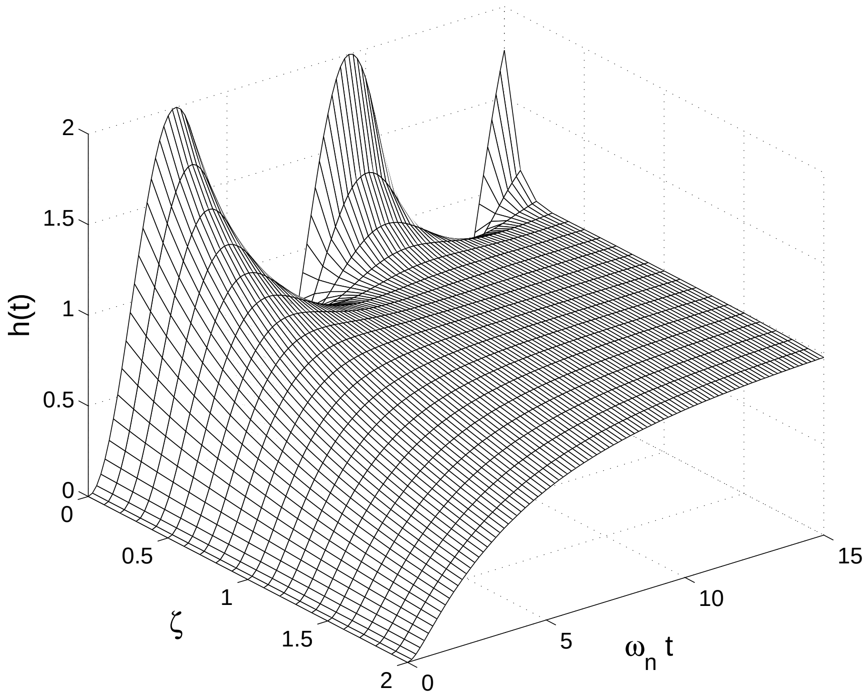


(a)

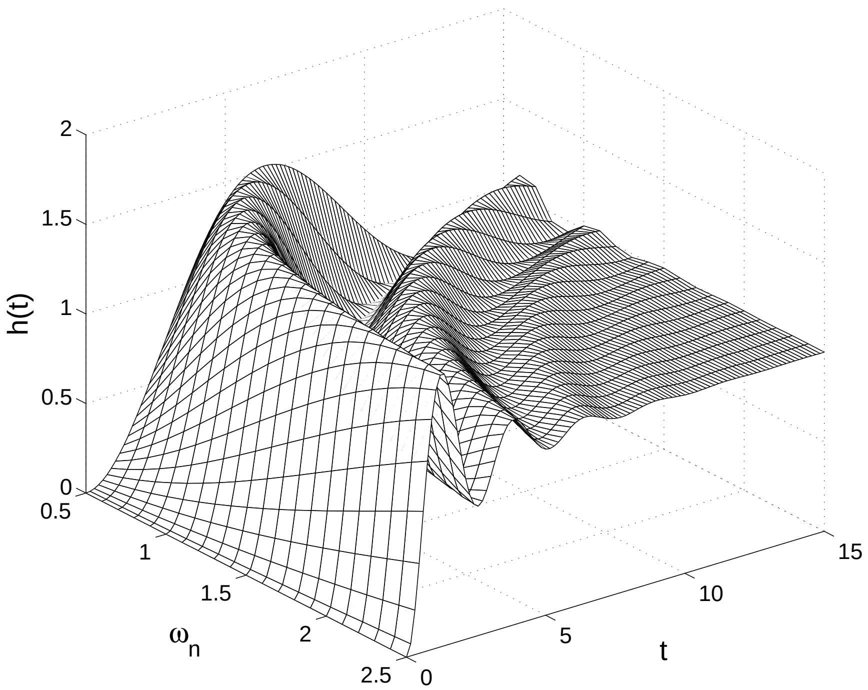


(b)

Rys. 24.



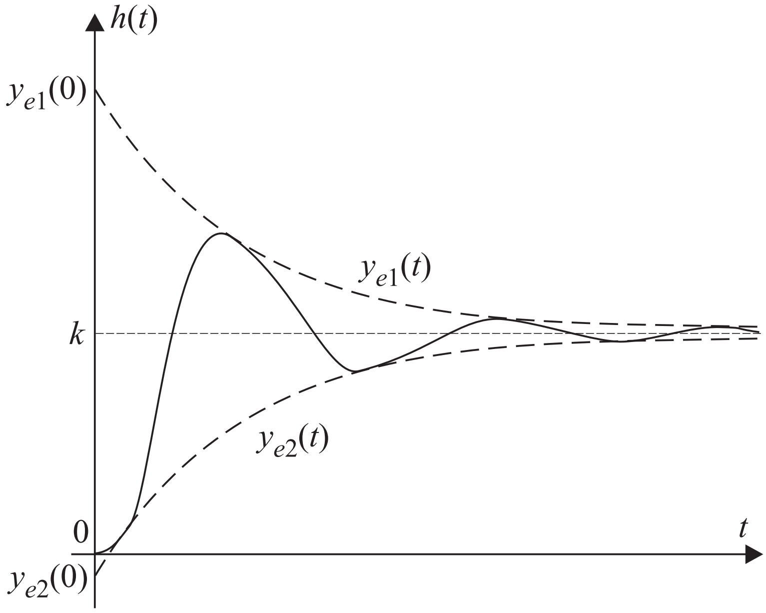
Rys. 25. $k = 1$



Rys. 26. $k = 1$, $\zeta = 0{,}2$

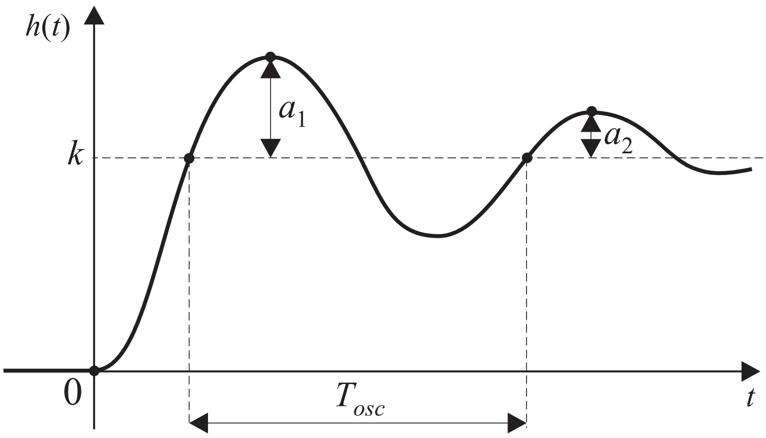
przyjmując $\sin(\cdot) = \pm 1$ otrzymujemy z $h(t)$ równanie obwiedni:

$$y_{e1/2}(t) = k\left(1 \pm \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\mathbb{1}(t) \tag{30}$$



Rys. 27.

$$y_{e1/2}(0) = k\left(1 \pm \frac{1}{\sqrt{1-\zeta^2}}\right), \qquad T_e = \frac{1}{\zeta\omega_n}$$



Rys. 28. $\zeta = ?$, $T_n = ?$

$$\sqrt{1-\zeta^2}/T_n = 2\pi/T_{osc}, \qquad \zeta/T_n = \left(\ln\frac{a_1}{a_2}\right)/T_{osc} \tag{31}$$

11. Charakterystyka amplitudowo-fazowa elementu oscylacyjnego

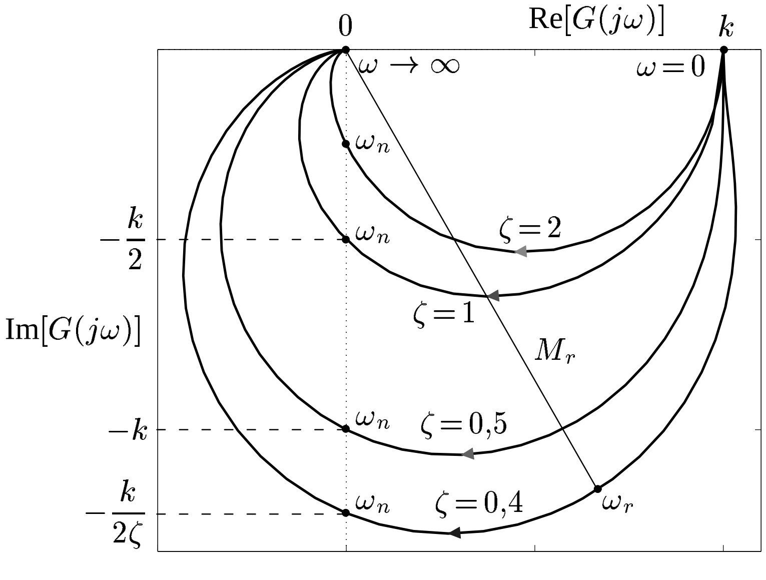
$$G(j\omega) = \frac{k\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} = \frac{k\omega_n^2}{(\omega_n^2 - \omega^2) + j\cdot 2\zeta\omega_n\omega} =$$

$$= k\omega_n^2 \frac{(\omega_n^2 - \omega^2) - j\cdot 2\zeta\omega_n\omega}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2} \tag{32}$$

zatem $Q(\omega) < 0$, zaś $P(\omega) > 0$ dla $\omega < \omega_n$ lub $P(\omega) < 0$ dla $\omega > \omega_n$

$$G(j\omega) = \left|G(j\omega)\right| e^{j\varphi(\omega)} \tag{33}$$

$$\left|G(j\omega)\right| = \frac{k\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}}, \qquad \varphi(\omega) = -\mathrm{arctg}\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}$$



Rys. 29.

$$K(\omega) = k^2\omega_n^4/\left|G(j\omega)\right|^2 = (\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2$$

$$\frac{dK(\omega)}{d\omega} = 2(\omega_n^2 - \omega^2)(-2\omega) + 8\zeta^2\omega_n^2\omega =$$

$$= -4\omega_n^2\omega + 4\omega^3 + 8\zeta^2\omega_n^2\omega = 4\omega\left(\omega^2 + 2\zeta^2\omega_n^2 - \omega_n^2\right)$$

$$\omega = 0 \quad \vee \quad \omega^2 = \omega_n^2(1 - 2\zeta^2) \to \omega_r = \omega_n\sqrt{1 - 2\zeta^2}$$

przy puls. rezonansowej $\omega_r = \omega_n\sqrt{1-2\zeta^2}$ moduł $|G(j\omega)|$ osiąga maksimum:

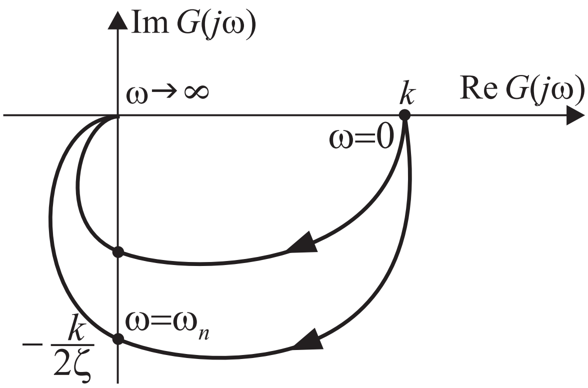
$$M_r = \max_{\omega}\big|G(j\omega)\big| = \big|G(j\omega_r)\big| = \frac{k\omega_n^2}{\sqrt{[\omega_n^2-\omega_n^2(1-2\zeta^2)]^2+4\zeta^2\omega_n^4(1-2\zeta^2)}} =$$

$$= \frac{k\omega_n^2}{\sqrt{4\omega_n^4\zeta^4+4\omega_n^4\zeta^2-8\omega_n^4\zeta^4}} = \frac{k\omega_n^2}{\sqrt{4\omega_n^4\zeta^2-4\omega_n^4\zeta^4}} = \frac{k}{2\zeta\sqrt{1-\zeta^2}} \quad (34)$$

$$\varphi(\omega_r) = -\mathrm{arctg}\frac{2\zeta\omega_n\omega_r}{\omega_n^2-\omega_r^2} = -\mathrm{arctg}\frac{\sqrt{1-2\zeta^2}}{\zeta} \quad (35)$$

$\omega_r = \omega_n\sqrt{1-2\zeta^2} \;\rightarrow\;$ rezonans występuje jedynie dla $0 \leqslant \zeta < \dfrac{\sqrt{2}}{2}$

$$Q(\omega_n) = \mathrm{Im}[G(\omega_n)] = -\frac{k}{2\zeta} \Rightarrow \begin{cases} |Q(\omega_n)| \leqslant \frac{1}{2}P(0) & \text{inerc. 2-go rz.} \\ |Q(\omega_n)| > \frac{1}{2}P(0) & \text{oscylacyjny} \end{cases}$$



Rys. 30.

12. Charakterystyki logarytmiczne elementu oscylacyjnego

$$\mathrm{Lm}(\omega) = 20\lg|G(j\omega)| = 20\lg(k\omega_n^2) - 20\lg\sqrt{(\omega_n^2-\omega^2)^2 + 4\zeta^2\omega_n^2\omega^2} =$$

$$= \begin{cases} 20\lg k + 20\lg\omega_n^2 - 20\lg\omega_n^2 & \text{dla } \omega \ll \omega_n \\ 20\lg k + 20\lg\omega_n^2 - 20\lg\omega^2 & \text{dla } \omega \gg \omega_n \end{cases} =$$

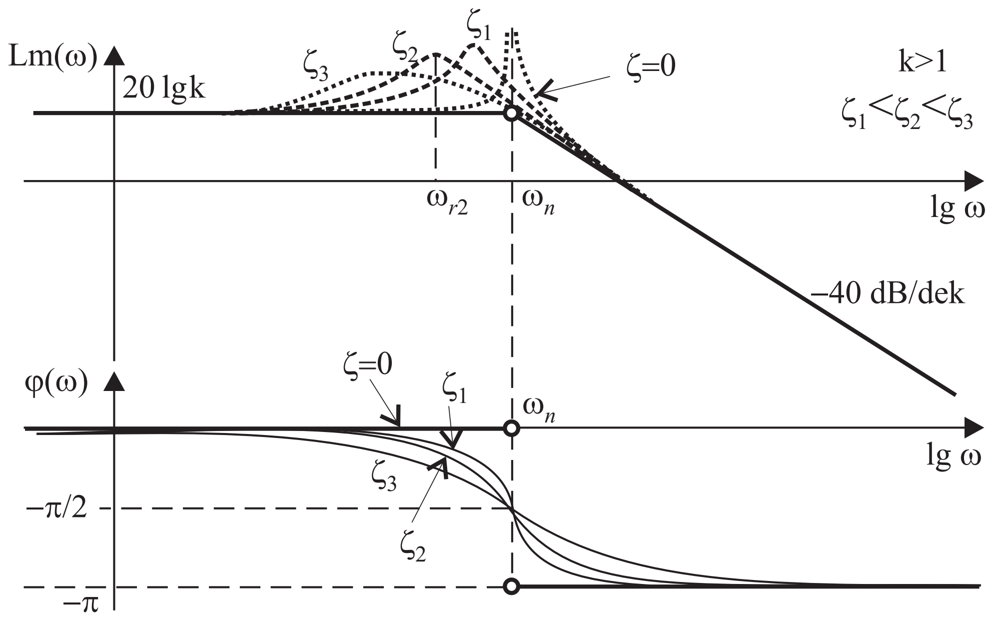
$$= \begin{cases} 20\lg k & \text{dla } \omega \ll \omega_n \\ 20\lg k - 40\lg\dfrac{\omega}{\omega_n} & \text{dla } \omega \gg \omega_n \end{cases} \qquad (k>0) \tag{36}$$

nachylenie $\mathrm{Lm}(\omega)$ dla $\omega \gg \omega_n$ wynosi $-40\left[\frac{\mathrm{dB}}{\mathrm{dek}}\right]$

$$\mathrm{Lm}(\omega_r) = 20\lg k - 20\lg(2\zeta\sqrt{1-\zeta^2}) \tag{37}$$

czyli $\quad \zeta \searrow \quad \Rightarrow \quad \mathrm{Lm}(\omega_r) \nearrow$

$$\varphi(\omega) = -\mathrm{arctg}\frac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}$$

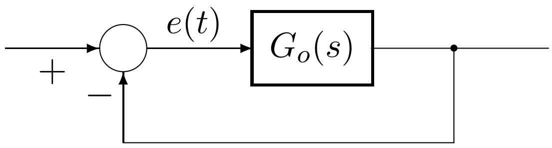


Rys. 31.

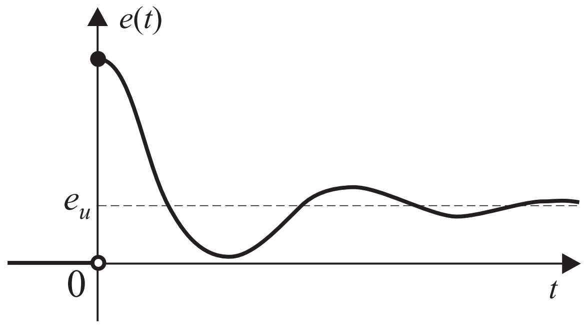
13. Wskaźniki czasowe jakości regulacji

a) uchyb ustalony e_u
b) czas opóźnienia t_d
c) czas narastania t_r
d) czas osiągnięcia wartości maksymalnej t_p
e) przeregulowanie M_p lub względne przeregulowanie $\varkappa$
f) czas ustalania (regulacji) t_s

Ad. a) uchyb ustalony e_u



(a)



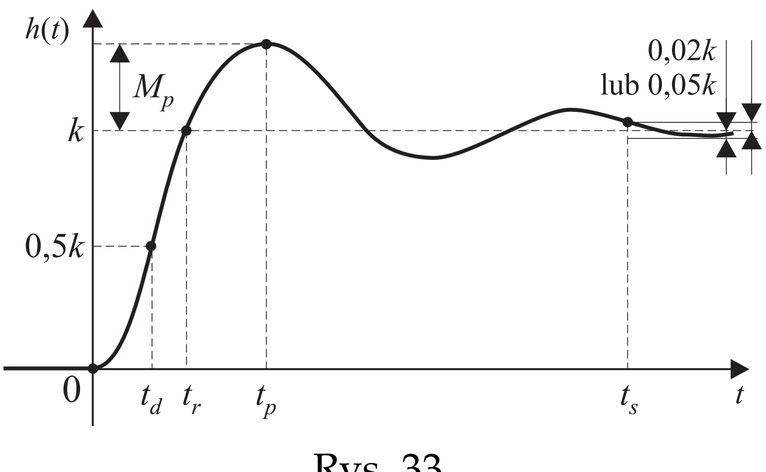
(b)

Rys. 32.

$$e(t) = e_u + e_p(t) \tag{38}$$

$$\lim_{t\to\infty} e_p(t) = 0 \quad \Rightarrow \quad \lim_{t\to\infty} e(t) = e_u = \lim_{s\to 0} sE(s) \tag{39}$$

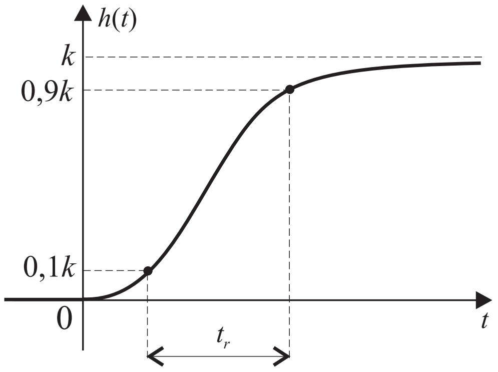
Ad. b) czas opóźnienia t_d



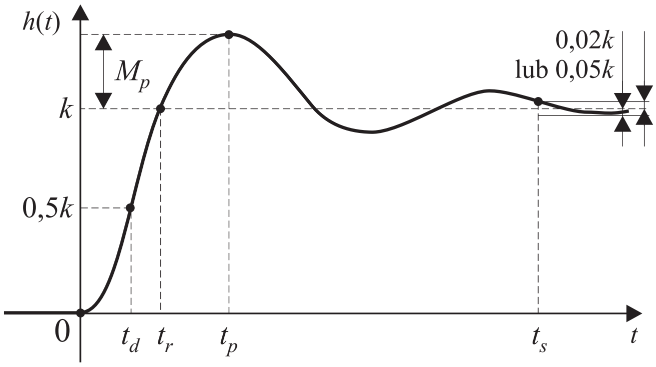
Rys. 33.

Ad. c) czas narastania t_r

$$10\% \to 90\% \quad \text{wartości ustalonej dla} \quad \zeta \geqslant 1$$
$$0\% \to 100\% \quad \text{wartości ustalonej dla} \quad 0 < \zeta < 1$$



(a)



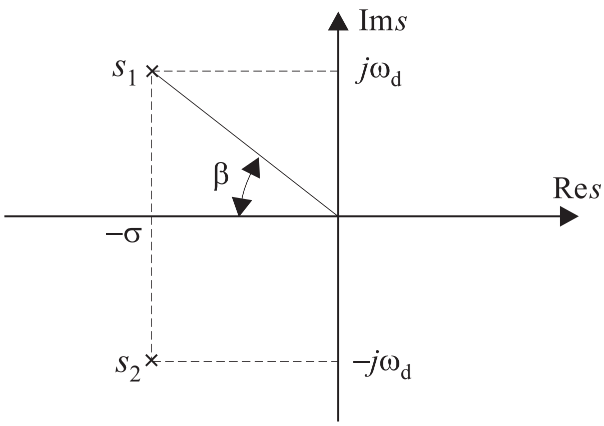
Rys. 34. (b)

$$h(t_r) = k = k\left[1 - e^{-\zeta\omega_n t_r}\left(\cos(\omega_d t_r) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r)\right)\right]$$

$$e^{-\zeta\omega_n t_r} \neq 0 \quad \Rightarrow \quad \cos(\omega_d t_r) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r) = 0$$

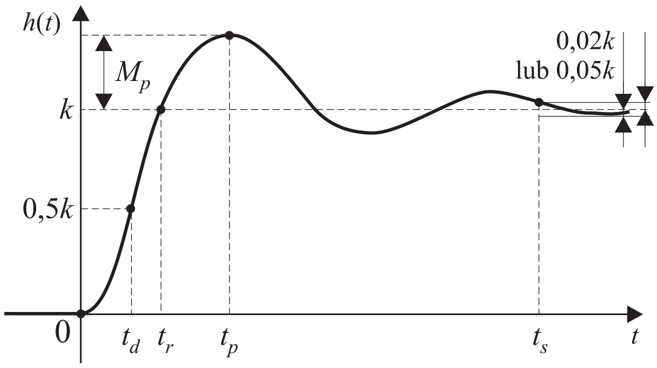
$$\mathrm{tg}(\omega_d t_r) = -\frac{\sqrt{1-\zeta^2}}{\zeta} = -\frac{\omega_d}{\sigma}, \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}, \quad \sigma = \omega_n\zeta$$

$$\boxed{t_r = \frac{1}{\omega_d}\mathrm{arctg}\frac{\omega_d}{-\sigma} = \frac{\pi - \beta}{\omega_d}} \qquad (40)$$



Rys. 35. $s_1 = -\sigma + j\omega_d, \quad s_2 = -\sigma - j\omega_d$

Ad. d) czas osiągnięcia wartości maksymalnej t_p



Rys. 36.

$$h(t) = k\left[1 - e^{-\zeta\omega_n t}\left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right)\right]$$

$$\frac{dh(t)}{dt} = k\zeta\omega_n e^{-\zeta\omega_n t}\left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right) +$$

$$+\, ke^{-\zeta\omega_n t}\left(\omega_d \sin(\omega_d t) - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos(\omega_d t)\right) =$$

$$= ke^{-\zeta\omega_n t}\left[\zeta\omega_n\cos(\omega_d t) + \frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\sin(\omega_d t) + \omega_n\sqrt{1-\zeta^2}\sin(\omega_d t) - \right.$$

$$\left. -\zeta\omega_n\cos(\omega_d t)\right] = k\omega_n\sin(\omega_d t)e^{-\zeta\omega_n t}\left(\frac{\zeta^2}{\sqrt{1-\zeta^2}} + \frac{1-\zeta^2}{\sqrt{1-\zeta^2}}\right)$$

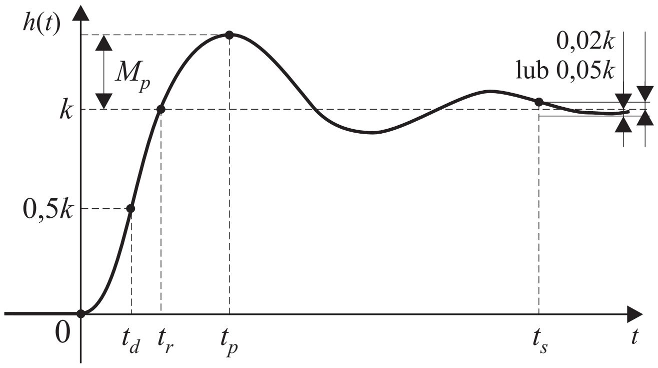
$$\left.\frac{dh(t)}{dt}\right|_{t=t_p} = k\sin(\omega_d t_p)\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t_p} = 0 \qquad (41)$$

czyli $\sin(\omega_d t_p) = 0$ lub $\omega_d t_p = 0, \pi, 2\pi, \ldots$

czas t_p odpowiada pierwszemu przeregulowaniu $\omega_d t_p = \pi$:

$$\boxed{t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}} \qquad (42)$$

Ad. e) przeregulowanie M_p lub względne przeregulowanie $\varkappa$



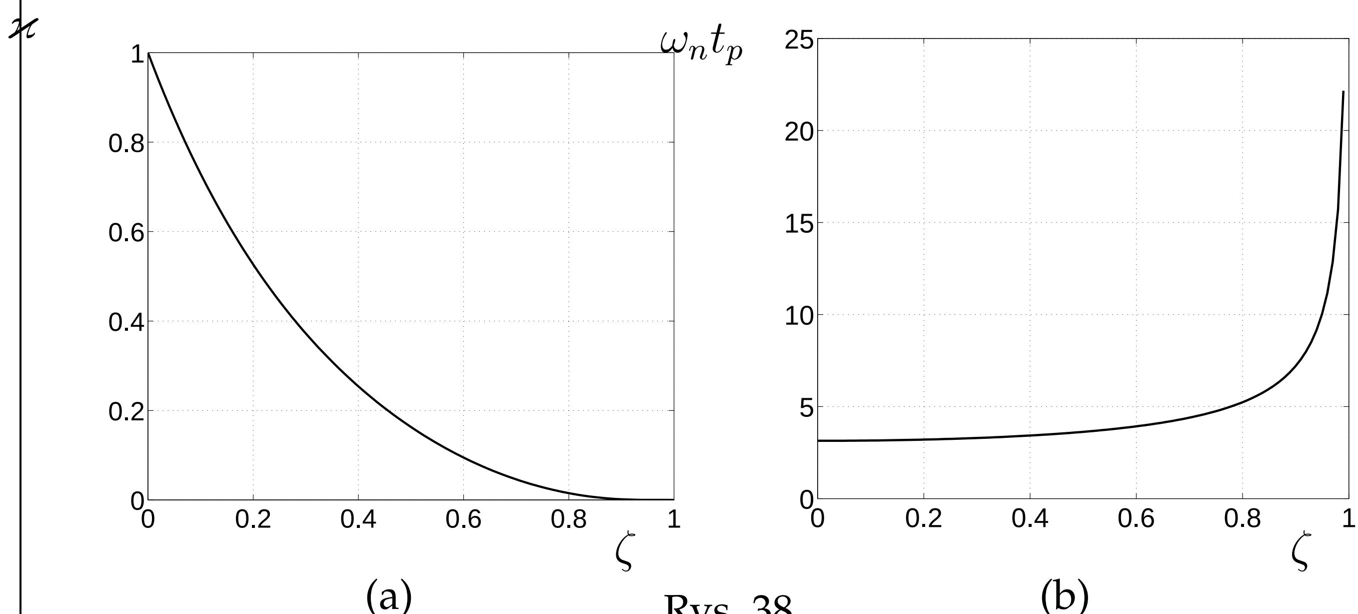
Rys. 37.

$$M_p = h(t_p) - h(\infty) = k\left[1 - e^{-\zeta\omega_n t_p}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right)\right] - k =$$

$$= -ke^{-\zeta\omega_n(\pi/\omega_d)}\left(\underbrace{\cos\pi}_{=-1} + \frac{\zeta}{\sqrt{1-\zeta^2}}\underbrace{\sin\pi}_{=0}\right) = ke^{-\frac{\sigma}{\omega_d}\pi} = ke^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}$$

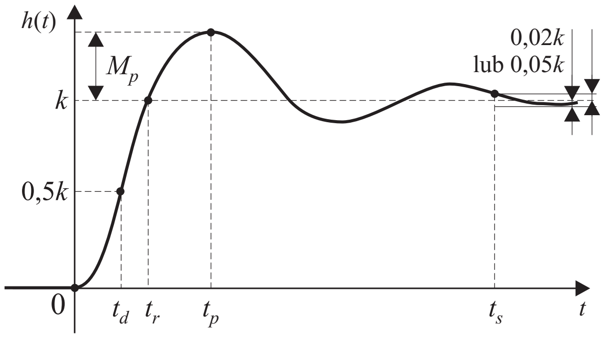
$$\boxed{M_p = ke^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}} \tag{43}$$

$$\boxed{\varkappa = \frac{M_p}{h(\infty)} = \frac{h(t_p) - h(\infty)}{h(\infty)} = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}} \tag{44}$$

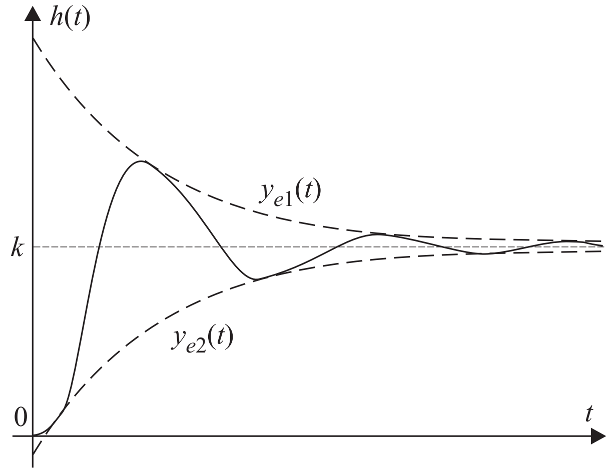


Rys. 38.

Ad. f) czas ustalania (regulacji) t_s



(a)

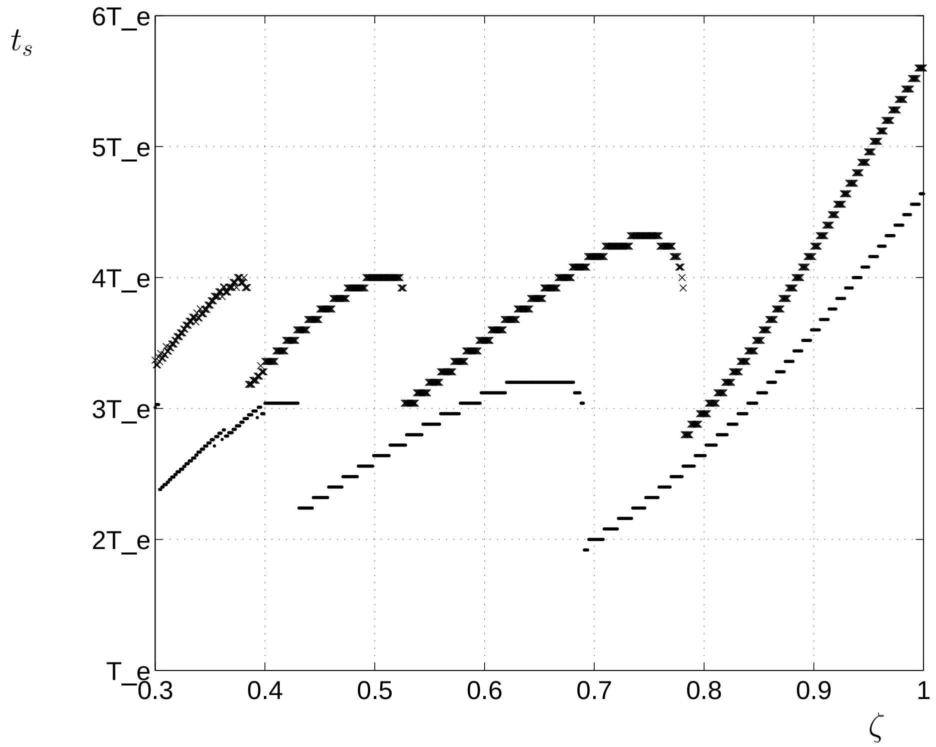


(b)

Rys. 39.

$$y_{e1/2}(t) = k\left(1 \pm \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\mathbb{1}(t), \qquad T_e = \frac{1}{\zeta\omega_n}$$

Przyjmuje się, że $t_s = 4T_e = 4/(\zeta\omega_n)$ dla kryterium $\pm 2\%$, $t_s = 3T_e = 3/(\zeta\omega_n)$ dla kryterium $\pm 5\%$



Rys. 40.