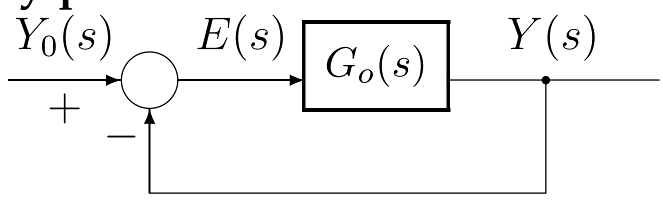
1. Kryterium Nyquista



Rys. 1

$$G_o(s)=\frac{Y(s)}{E(s)}=\frac{L_o(s)}{M_o(s)} \;\rightarrow G_z(s)=\frac{Y(s)}{Y_0(s)}=\frac{G_o(s)}{1+G_o(s)}=\frac{L_o(s)}{L_o(s)+M_o(s)} \quad (1)$$

równanie charakterystyczne ukł. otwartego: $M_o(s) = 0$;
równ. charakteryst. ukł. zamkniętego: $M_z(s) = L_o(s) + M_o(s) = 0$;
wielom. charakt. $M_z(s)$ ma te same miejsca zerowe, co wyrażenie:

$$1 + G_o(s) = \frac{L_o(s) + M_o(s)}{M_o(s)} \quad \text{czyli} \quad G_o(j\omega) = G_o(s)\Big|_{s=j\omega} = -1.$$

Jeśli $G(j\omega) = |G(j\omega)|e^{j\varphi(\omega)}$, to ozn. $\arg[G(j\omega)] = \varphi(\omega)$

$$\underset{0\leqslant\omega<\infty}{\Delta} \arg\,[1+G_o(j\omega)]=\underset{0\leqslant\omega<\infty}{\Delta} \arg\,[\underbrace{L_o(j\omega)+M_o(j\omega)}_{M_z(j\omega)}]-\underset{0\leqslant\omega<\infty}{\Delta} \arg\,[M_o(j\omega)] \quad (2)$$

Jeśli ukł. zamknięty jest stabilny, wówczas na podst. kryterium Michajłowa:

$$\underset{0\leqslant\omega<\infty}{\Delta} \arg\,[M_z(j\omega)] = \underset{0\leqslant\omega<\infty}{\Delta} \arg\,[L_o(j\omega) + M_o(j\omega)] = n\frac{\pi}{2} \quad (3)$$

($n =$ st. $M_z(s) =$ st. $M_o(s)$ gdyż st. $L_o(s) \leqslant$ st. $M_o(s)$)

Jeśli ukł. otwarty jest stabilny, to

$$\underset{0\leqslant\omega<\infty}{\Delta} \arg\,[M_o(j\omega)] = n\frac{\pi}{2}, \quad (4)$$

a jeśli niestabilny (l biegunów w prawej półpł. zesp.), to

$$\underset{0\leqslant\omega<\infty}{\Delta} \arg\,[M_o(j\omega)] = (n-2l)\frac{\pi}{2}. \quad (5)$$

$$A(j\omega) = P(\omega) + jQ(\omega) = |A(j\omega)|e^{j\varphi(\omega)}, \qquad \varphi(\omega) = \arg(A(j\omega))$$

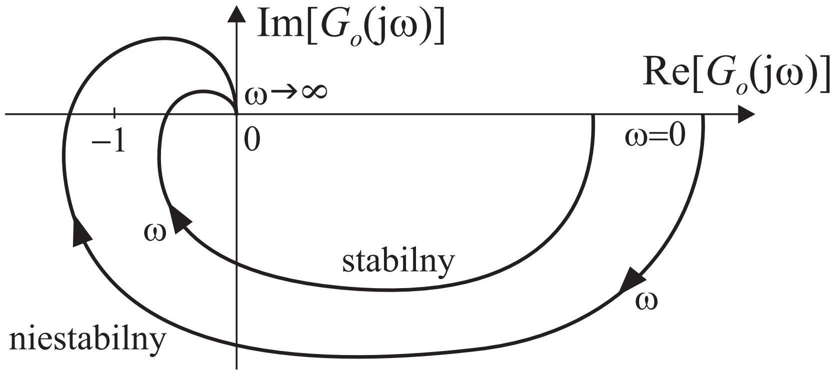
Jeśli ukł. otwarty jest stabilny, to ukł. zamkn. też stabilny jeśli

$$\underset{0\leqslant\omega<\infty}{\Delta}\arg[1 + G_o(j\omega)] = n\frac{\pi}{2} - n\frac{\pi}{2} = 0, \tag{6}$$

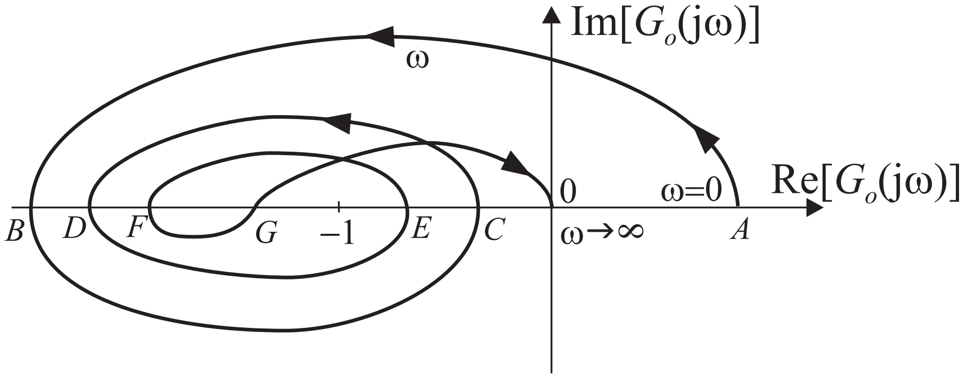
a jeśli ukł. otwarty jest niestabilny, to ukł. zamkn. stabilny jeśli

$$\underset{0\leqslant\omega<\infty}{\Delta}\arg[1 + G_o(j\omega)] = n\frac{\pi}{2} - (n-2l)\frac{\pi}{2} = l\pi = \frac{l}{2}\cdot 2\pi \tag{7}$$

$(-1, j0), \qquad l = k_1 + 2k_2 + 3k_3 + \ldots$



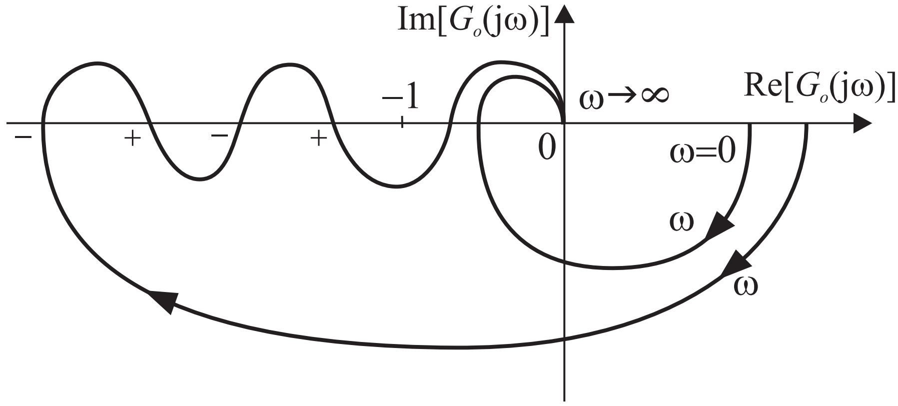
Rys. 2



Rys. 3

$$\left.\begin{array}{ll} AB:\ \Delta_1\varphi = \pi, & EF:\ \Delta_5\varphi = \pi, \\ BC:\ \Delta_2\varphi = \pi, & FG:\ \Delta_6\varphi = 0, \\ CD:\ \Delta_3\varphi = \pi, & G0:\ \Delta_7\varphi = -\pi, \\ DE:\ \Delta_4\varphi = \pi, & \end{array}\right\} \Delta\varphi = \sum \Delta_i\varphi = 4\pi$$

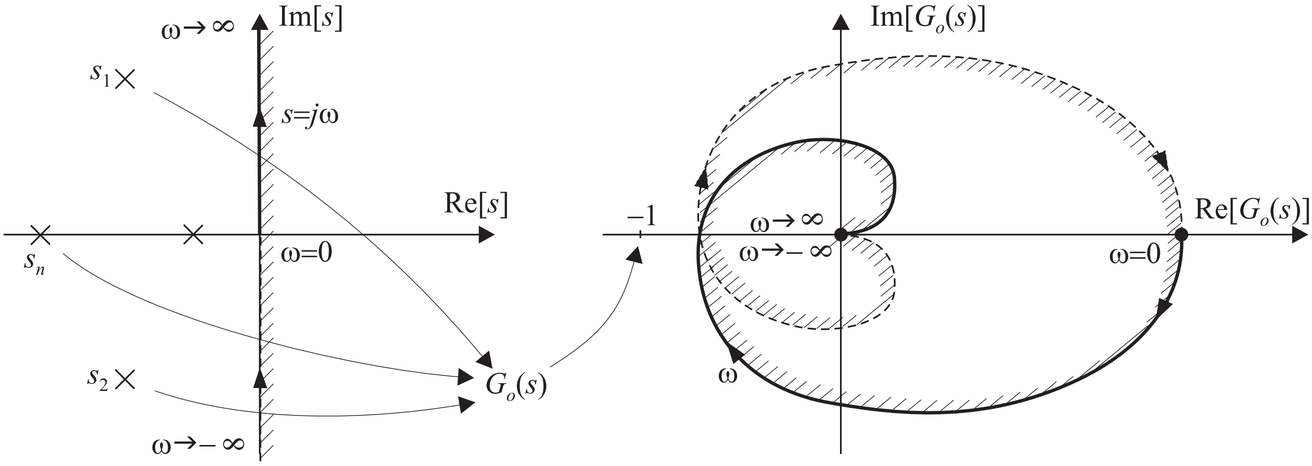
$\Delta\varphi = +2\pi$ między dwoma sąsiedn. przecięciami dodatnimi,
$\Delta\varphi = -2\pi$ między dwoma sąsiedn. przecięciami ujemnymi,
$\Delta\varphi = 0$ między sąsiedn. przecięciami dodatnim i ujemnym



Rys. 4 Układy stabilne po zamknięciu również stabilne

2. Odwzorowanie płaszczyzny zespolonej s na płaszczyznę zespoloną $G(s)$

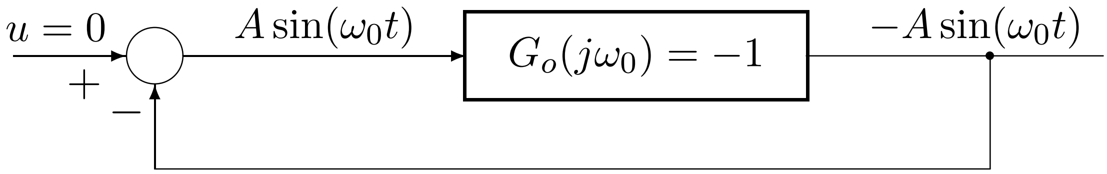
r. char. $M_z(s) = 0$ jest równoważne $1 + G_o(s) = 0$, tzn. $G_o(s) = -1$



Rys. 5

bieguny transm. ukł. zamkniętego $s_1, s_2, \ldots s_n$ (pierwiastki równ. char. $M_z(s) = L_o(s) + M_o(s) = 0$) $\to G_o(s) = -1$

3. Kryterium Nyquista – podejście częstotliwościowe



Rys. 6

4. Kryterium Nyquista dla układu otwartego na granicy stabilności

$$G_o(s) = k_o \frac{(s-\sigma_1)(s-\sigma_2)\dots(s-\sigma_m)}{s(s-s_1)(s-s_2)\dots(s-s_{n-1})} \tag{8}$$

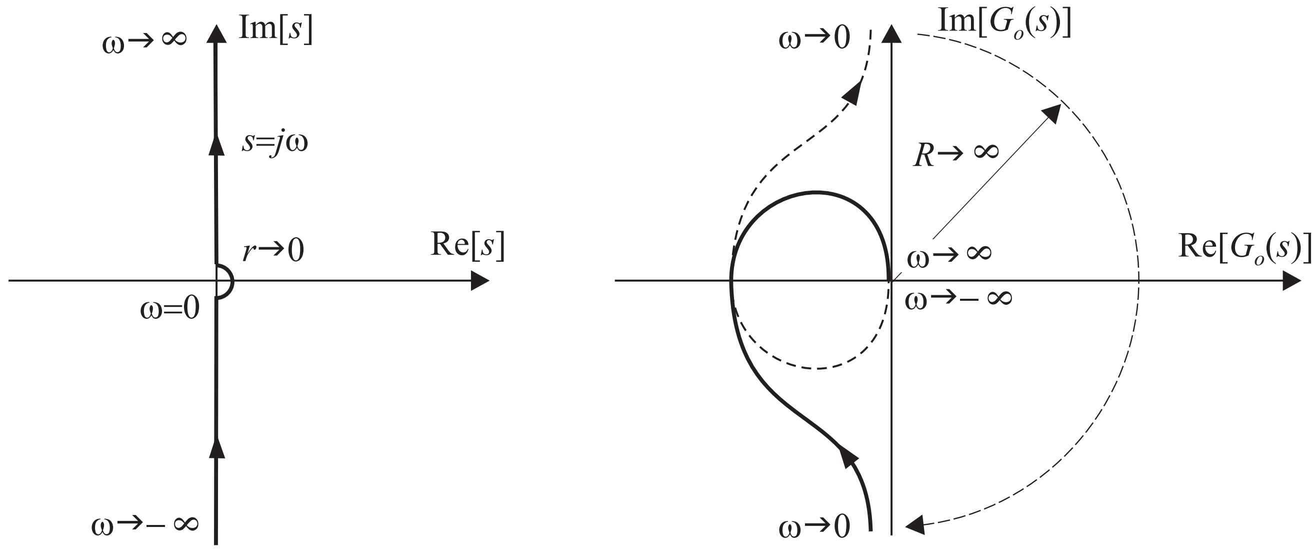
$m \leqslant n$, $\mathrm{Re}(s_i) < 0$, $i = 1, 2, \dots n-1$ oraz $\sigma_j \neq 0$, $j = 1, 2, \dots m$

$$s = re^{j\varphi}, \qquad \varphi:\ -\frac{\pi}{2} \to \frac{\pi}{2}, \qquad r \to 0,\ r > 0 \tag{9}$$

$$s \to 0 \quad \Rightarrow \quad G_o(s) \to \frac{B_o}{s}, \quad B_o = k_o \left(\prod_{j=1}^{m}(-\sigma_j)\right) \Big/ \left(\prod_{i=1}^{n-1}(-s_i)\right)$$

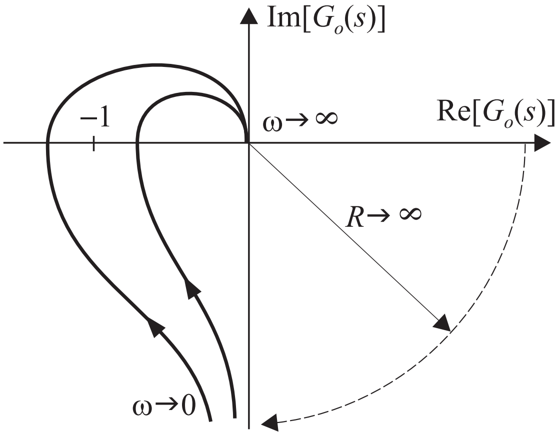
wtedy

$$G_o(s) \approx \frac{B_o}{s} = \frac{B_o}{re^{j\varphi}} = Re^{-j\varphi}, \quad R \to \infty,\ -\varphi:\ \frac{\pi}{2} \to -\frac{\pi}{2} \tag{10}$$

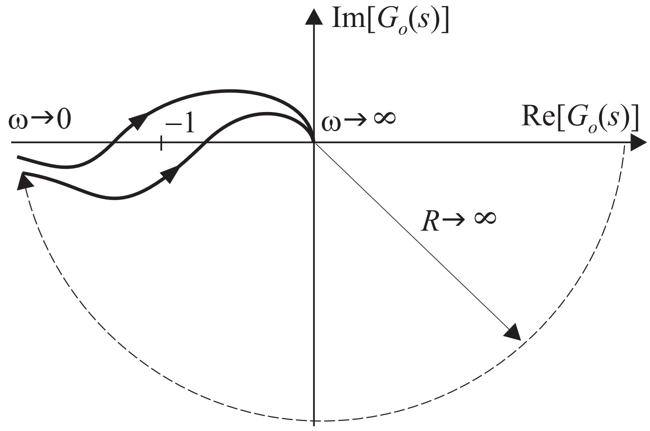


Rys. 7

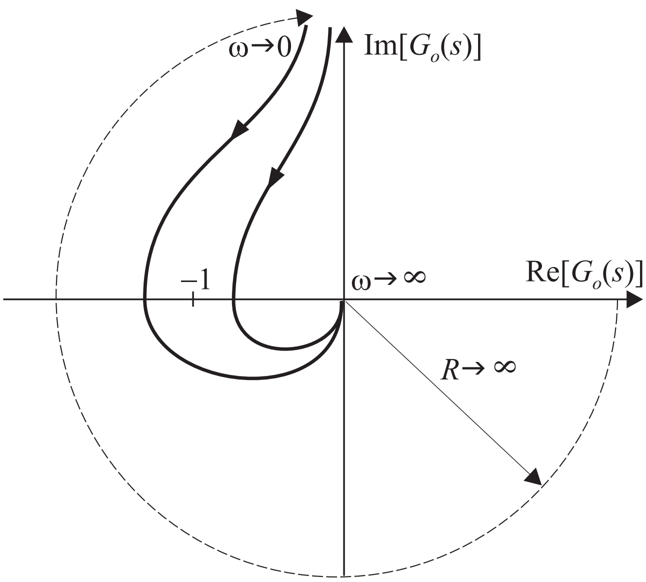
$$G_o(s) = \frac{L(s)}{s^\nu M_1(s)} \tag{11}$$



(a) $\nu = 1$



(b) $\nu = 2$



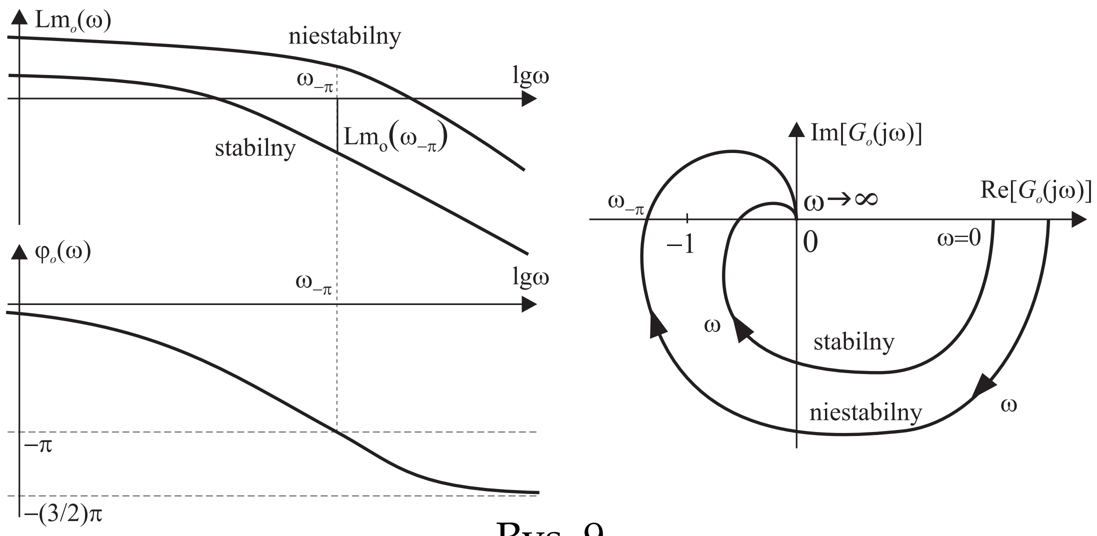
(c) $\nu = 3$

Rys. 8

5. Logarytmiczne kryterium Nyquista

$$\mathrm{Lm}_o(\omega) = 20\lg|G_o(j\omega)|, \qquad \varphi_o(\omega) = \arg(G_o(j\omega))$$

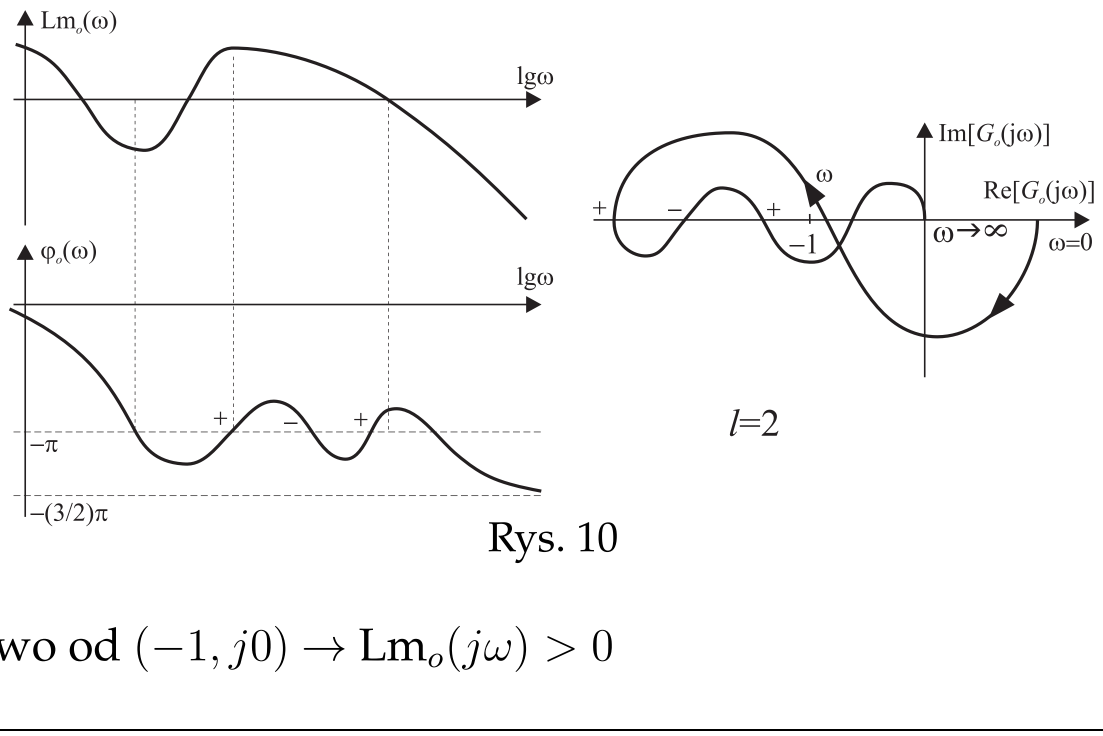
a) układ otwarty stabilny asymptotycznie



Rys. 9

$$\begin{aligned} \mathrm{Lm}_o(\omega_{-\pi}) < 0 \quad &\Leftrightarrow \quad 0 < |G_o(j\omega_{-\pi})| < 1 \\ \mathrm{Lm}_o(\omega_{-\pi}) = 0 \quad &\Leftrightarrow \quad |G_o(j\omega_{-\pi})| = 1 \\ \mathrm{Lm}_o(\omega_{-\pi}) > 0 \quad &\Leftrightarrow \quad |G_o(j\omega_{-\pi})| > 1 \end{aligned} \tag{12}$$

b) układ otwarty niestabilny (l biegunów w prawej półpł.)



Rys. 10

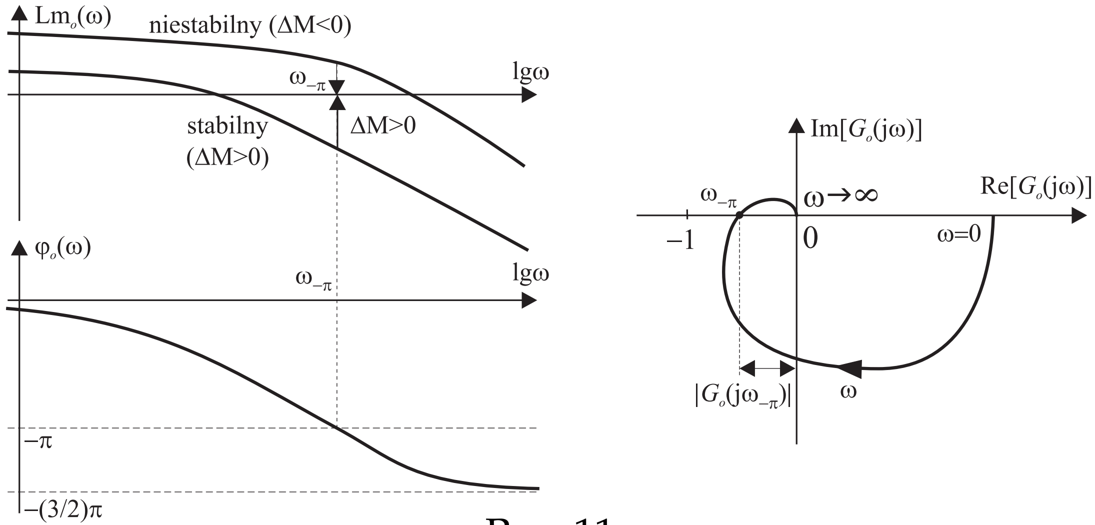
na lewo od $(-1, j0) \rightarrow \mathrm{Lm}_o(j\omega) > 0$

6. Zapasy stabilności

Zapas modułu

$$\text{liniowy:} \quad \lambda = \frac{1}{|G_o(j\omega_{-\pi})|} \tag{13}$$

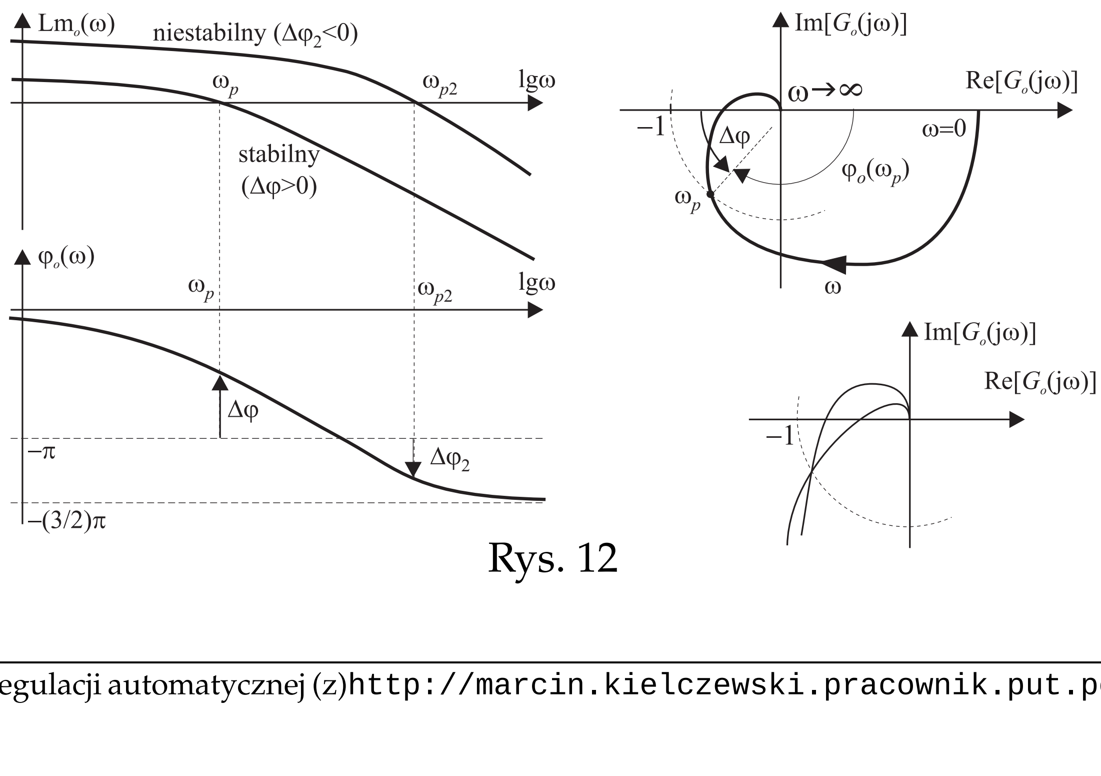
$$\text{logarytmiczny:} \quad \Delta M = 20 \lg \lambda = -20 \lg |G_o(j\omega_{-\pi})| \tag{14}$$



Rys. 11

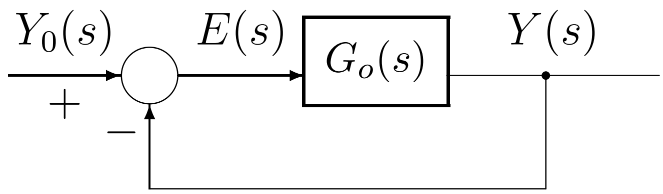
Zapas fazy

$$-\pi + \Delta\varphi = \varphi_o(\omega_p) \quad \Rightarrow \quad \Delta\varphi = \varphi_o(\omega_p) + \pi \tag{15}$$



Rys. 12

Przykład $\quad G_o(s) = \dfrac{k}{s(Ts+1)^2}, \qquad T = 0{,}1[\mathrm{s}]$



Rys. 13 $\qquad \Delta M \geqslant 8[\mathrm{dB}], \quad \Delta\varphi \geqslant 30^\circ = \pi/6$

$$|G_o(j\omega)| = \frac{k}{\omega(\omega^2T^2+1)}, \qquad \varphi_o(\omega) = -\frac{\pi}{2} - 2\mathrm{arctg}\omega T$$

Zapas modułu:

$$\begin{cases} \arg G_o(j\omega_{-\pi}) = -\pi \\ -20\lg|G_o(j\omega_{-\pi})| \geqslant \Delta M = 8[\mathrm{dB}] \quad (20\lg|G_o(j\omega_{-\pi})| \leqslant -\Delta M) \end{cases}$$

$$-\frac{\pi}{2} - 2\mathrm{arctg}\omega T = -\pi \ \rightarrow \ -\mathrm{arctg}\omega T = -\frac{\pi}{4} \ \rightarrow \ \omega_{-\pi} = \frac{1}{T} = 10[\mathrm{rad/s}]$$

$$20\lg|G_o(j\omega_{-\pi})| \leqslant -8 \ \rightarrow \ \lg|G_o(j\omega_{-\pi})| \leqslant -0{,}4$$

$$|G_o(j\omega_{-\pi})| \leqslant 10^{-0{,}4} = 0{,}398 \approx 0{,}4$$

$$|G_o(j\omega_{-\pi})| = \frac{k}{\omega_{-\pi}(\omega_{-\pi}^2T^2+1)} = \frac{k}{10(1+1)} = \frac{k}{20} \leqslant 0{,}4 \ \rightarrow \ k \leqslant 8$$

Zapas fazy:

$$\begin{cases} \arg G_o(j\omega_p) = -\pi + \Delta\varphi \\ 20\lg|G_o(j\omega_p)| \leqslant 0 \qquad (|G_o(j\omega_p)| \leqslant 1) \end{cases}$$

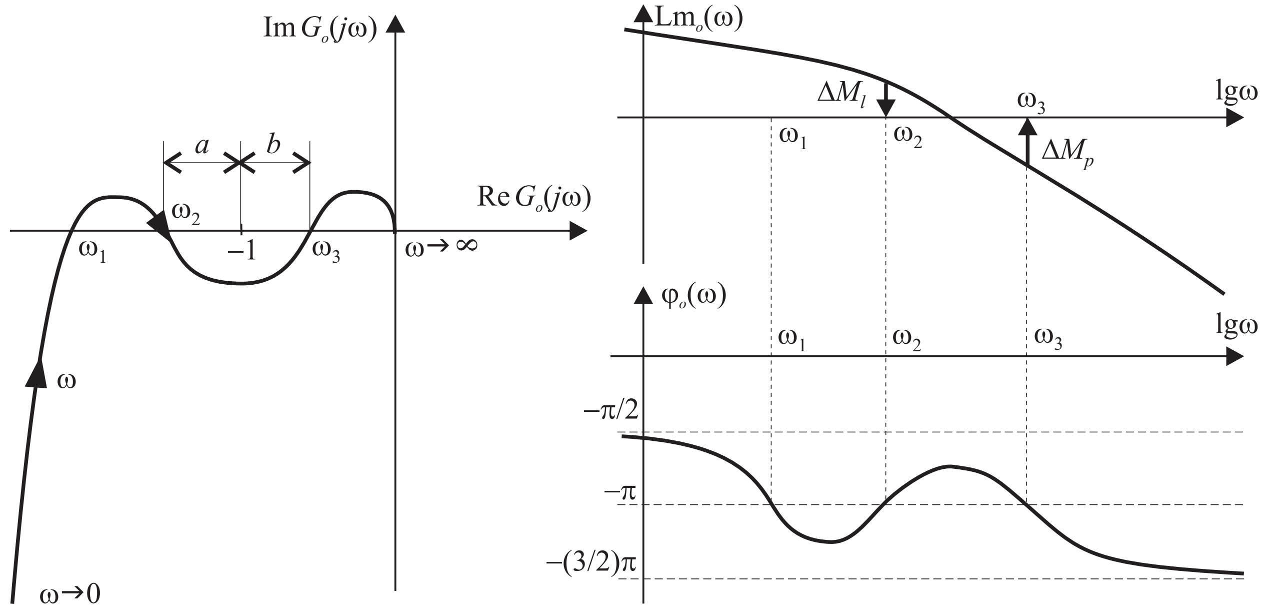
$$-\pi/2 - 2\mathrm{arctg}\omega T = -\pi + \pi/6 = -5\pi/6 \ \rightarrow \ -2\mathrm{arctg}\omega T = -\pi/3$$

$$\mathrm{arctg}\omega T = \frac{\pi}{6} \ \rightarrow \ \omega_p T = \sqrt{3}/3 \ \rightarrow \ \omega_p = 10\sqrt{3}/3$$

$$|G_o(j\omega_p)| = \frac{k}{\omega_p(\omega_p^2T^2+1)} = \frac{k}{10\sqrt{3}/3(1/3+1)} = \frac{9k}{40\sqrt{3}} \leqslant 1$$

$$k \leqslant 40\sqrt{3}/9 \approx 7{,}69$$

7. Układy warunkowo stabilne

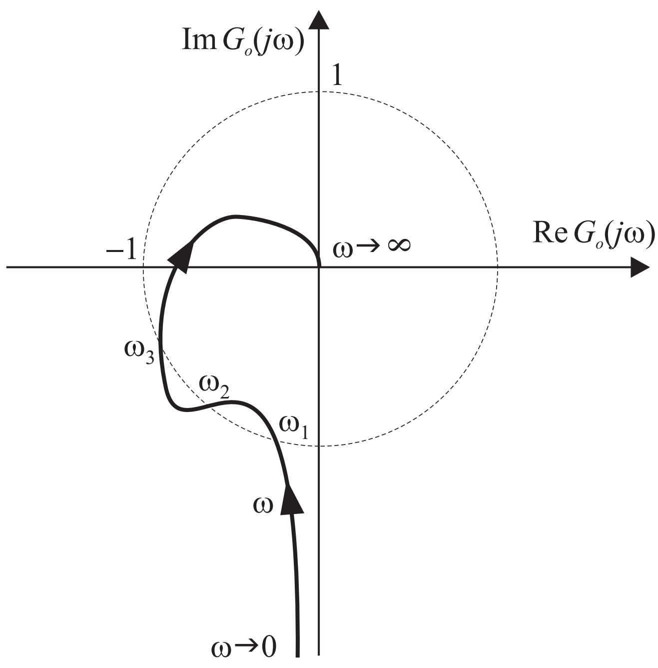


Rys. 14

$$\lambda_l = 1/a, \qquad \lambda_p = 1/b,$$

$$\Delta M_l = 20\lg\lambda_l = -20\lg a, \qquad \Delta M_p = 20\lg\lambda_p = -20\lg b$$

układ stabilny: $k \in (0, k_1) \cup (k_2, k_3)$



Rys. 15

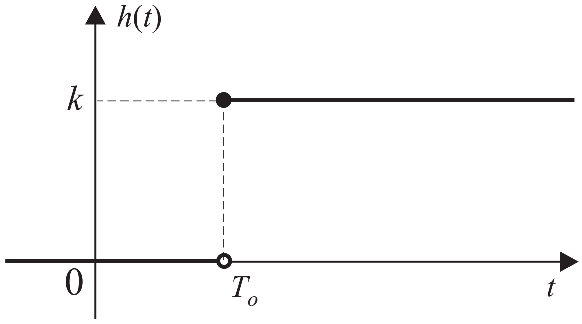
8. Element opóźniający

$$G(s) = \frac{Y(s)}{U(s)} = ke^{-sT_o}, \quad k > 0 \quad \rightarrow \tag{16}$$

$$\rightarrow \quad y(t) = \mathcal{L}^{-1}\left[G(s)U(s)\right] = k\,u(t - T_o)$$

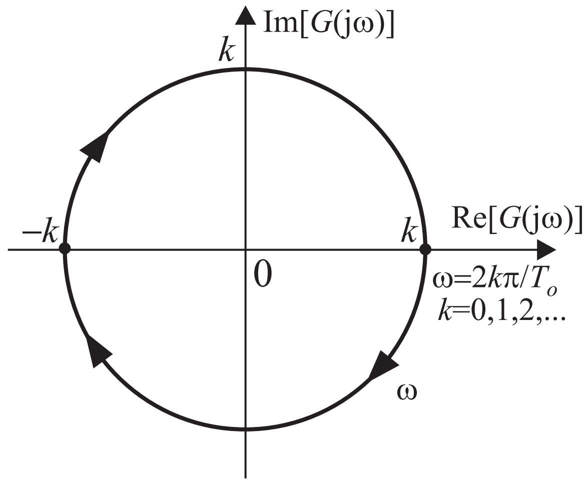
$$h(t) = \mathcal{L}^{-1}\left[\frac{G(s)}{s}\right] = k\,\mathbb{1}(t - T_o), \quad g(t) = \mathcal{L}^{-1}\left[G(s)\right] = k\,\delta(t - T_o)$$

$$G(j\omega) = ke^{-j\omega T_o} = k[\cos(\omega T_o) - j\sin(\omega T_o)] \tag{17}$$

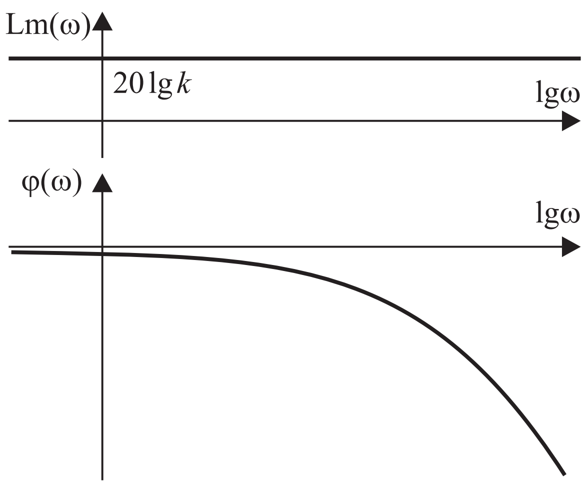


Rys. 16

$$\mathrm{Lm}(\omega) = 20\lg k + \underbrace{20\lg\left|e^{-j\omega T_o}\right|}_{=0} = 20\lg k, \qquad \varphi(\omega) = -\omega T_o \tag{18}$$



(a)



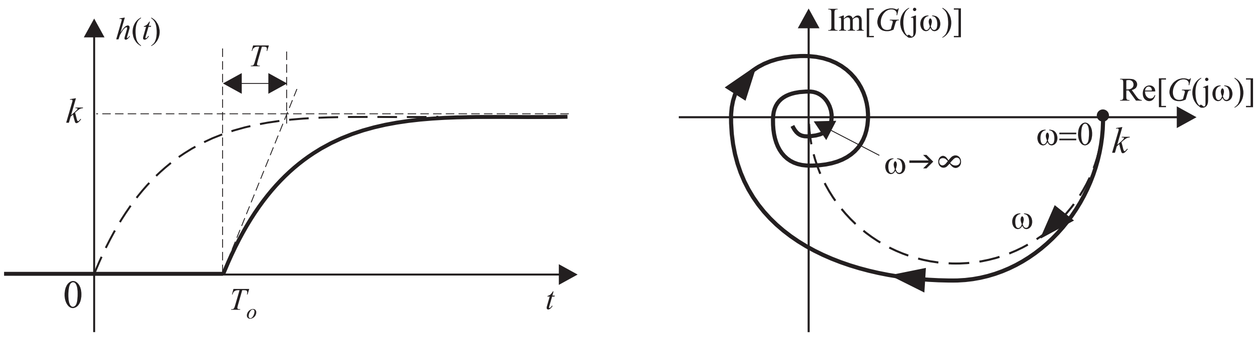
(b)

Rys. 17

Przykład (element inercyjny 1-go rzędu z opóźnieniem)

$$G(s) = \frac{k}{1+sT}e^{-sT_o}, \quad k > 0 \quad \rightarrow \quad h(t) = k\left(1 - e^{-\frac{t-T_o}{T}}\right)\mathbb{1}(t - T_o)$$

$$|G(j\omega)| = \frac{k}{\sqrt{1+\omega^2T^2}} \cdot 1, \qquad \varphi(\omega) = -\text{arctg}(\omega T) - \omega T_o$$



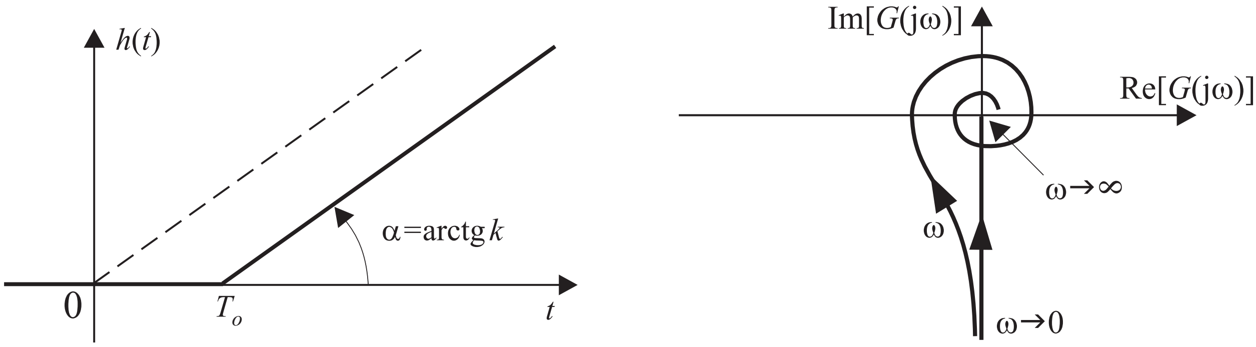
(a) (b)

Rys. 18

Przykład (element całkujący idealny z opóźnieniem)

$$G(s) = \frac{k}{s}e^{-sT_o}, \quad k > 0 \quad \rightarrow \quad h(t) = k(t - T_o)\mathbb{1}(t - T_o)$$

$$|G(j\omega)| = \frac{k}{\omega} \cdot 1, \qquad \varphi(\omega) = -\frac{\pi}{2} - \omega T_o$$

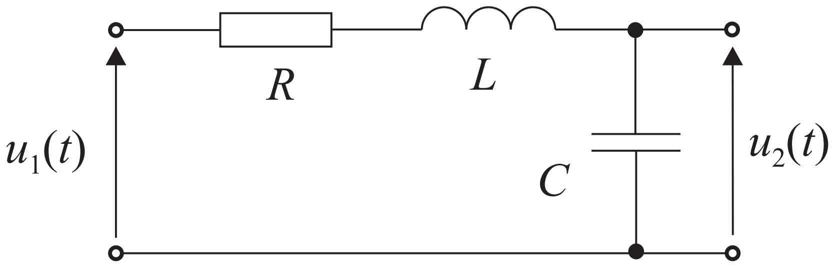


(a) (b)

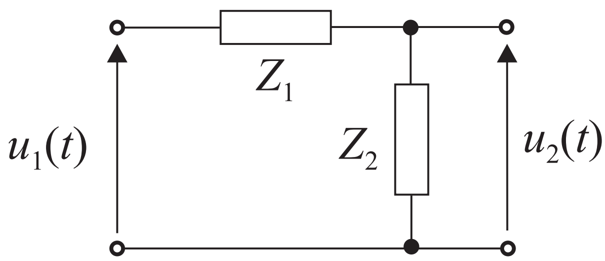
Rys. 19

9. Układy dynamiczne drugiego rzędu

Przykład



Rys. 20



Rys. 21

$$Z_1(s) = R + sL \qquad\qquad Z_2(s) = \frac{1}{sC}$$

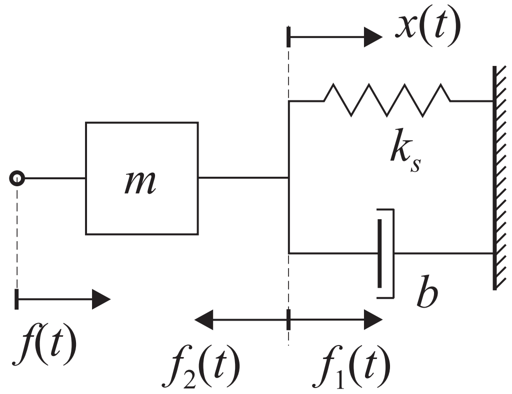
$$G(s) = \frac{U_2(s)}{U_1(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} = \frac{\dfrac{1}{sC}}{R + sL + \dfrac{1}{sC}} =$$

$$= \frac{1}{s^2LC + sRC + 1} = \frac{\dfrac{1}{LC}}{s^2 + s\dfrac{R}{L} + \dfrac{1}{LC}}$$

$$G(s) = \frac{k}{T_n^2 s^2 + 2\zeta T_n s + 1}$$

$$k = 1, \quad T_n = \sqrt{LC}, \quad 2\zeta T_n = RC \to \zeta = \frac{RC}{2T_n} = \frac{RC}{2\sqrt{LC}} = \frac{R}{2}\sqrt{\frac{C}{L}}$$

Przykład



Rys. 22

$$f(t) = m\frac{d^2x(t)}{dt^2} + f_1(t),$$
$$f_2(t) = k_s\left[0 - x(t)\right] + b\left[0 - \frac{dx(t)}{dt}\right],$$
$$f_1(t) + f_2(t) = 0,$$

$$f(t) = m\frac{d^2x(t)}{dt^2} - f_2(t) = m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + k_s x(t),$$
$$F(s) = (ms^2 + bs + k_s)X(s),$$

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k_s} = \frac{\frac{1}{k_s}}{\frac{m}{k_s}s^2 + \frac{b}{k_s}s + 1}$$

$$G(s) = \frac{k}{T_n^2 s^2 + 2\zeta T_n s + 1}$$

$$k = \frac{1}{k_s}, \quad T_n = \sqrt{\frac{m}{k_s}}$$

$$2\zeta T_n = \frac{b}{k_s} \rightarrow \zeta = \frac{b}{2k_s T_n} = \frac{b}{2k_s}\sqrt{\frac{k_s}{m}} = \frac{b}{2\sqrt{k_s m}}$$

$$G(s) = \frac{k}{T_n^2 s^2 + 2\zeta T_n s + 1} = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{19}$$

k wzmocnienie,

T_n stała czasowa,

$\omega_n = 1/T_n > 0$ pulsacja drgań nietłumionych (p. naturalna),

ζ współczynnik tłumienia (rozważamy $\zeta \geqslant 0$),

równanie charakterystyczne:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\Delta = 4\omega_n^2(\zeta^2 - 1)$$

a) <u>układ przetłumiony</u> (ang. overdamped): $\zeta > 1$, $\Delta > 0$

$$\sqrt{\Delta} = 2\omega_n\sqrt{\zeta^2 - 1}$$

$$s_{1/2} = \frac{-2\omega_n\zeta \pm 2\omega_n\sqrt{\zeta^2 - 1}}{2}$$

$$s_1 = -\omega_n(\zeta - \sqrt{\zeta^2 - 1}), \qquad s_2 = -\omega_n(\zeta + \sqrt{\zeta^2 - 1}), \tag{20}$$

$$s_1 s_2 = \omega_n^2\left(\zeta^2 - (\zeta^2 - 1)\right) = \omega_n^2$$

$$G(s) = \frac{k\omega_n^2}{(s - s_1)(s - s_2)} = \frac{k\omega_n^2}{s_1 s_2\left(-\frac{1}{s_1}s + 1\right)\left(-\frac{1}{s_2}s + 1\right)} =$$

$$= \frac{k}{(T_1 s + 1)(T_2 s + 1)}, \tag{21}$$

gdzie $T_1 = -\frac{1}{s_1}$, $T_2 = -\frac{1}{s_2}$, a wtedy $T_1 T_2 = \frac{1}{\omega_n^2}$, $T_1 + T_2 = \frac{2\zeta}{\omega_n}$

$$h(t) = k\left(1 - \frac{T_1 e^{-t/T_1} - T_2 e^{-t/T_2}}{T_1 - T_2}\right)\mathbb{1}(t)$$

b) układ tłumiony krytycznie: $\zeta = 1$, $\Delta = 0$

$$s_0 = -\frac{2\zeta\omega_n}{2} = -\zeta\omega_n = -\omega_n, \tag{22}$$

$$G(s) = \frac{k\omega_n^2}{(s-s_0)^2} = \frac{k\omega_n^2}{(s+\omega_n)^2} = \frac{k}{(Ts+1)^2}, \tag{23}$$

gdzie $T = -\dfrac{1}{s_0} = \dfrac{1}{\omega_n}$

$$h(t) = \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right] = \mathcal{L}^{-1}\left[\frac{k\omega_n^2}{(s+\omega_n)^2 s}\right] = k\left[1 - (1+\omega_n t)e^{-\omega_n t}\right]\mathbb{1}(t)$$

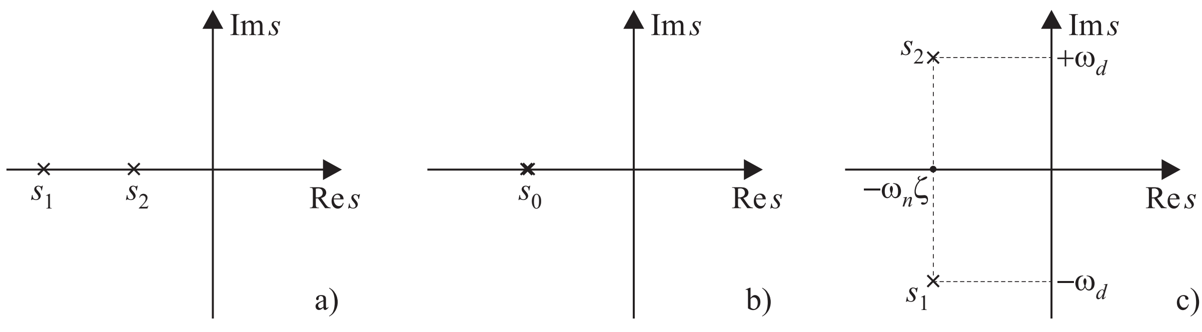
przebieg aperiodyczny krytyczny (24)

c) układ niedotłumiony (ang. underdamped): $0 \leqslant \zeta < 1$, $\Delta < 0$

$$\sqrt{\Delta} = \sqrt{4\omega_n^2(\zeta^2 - 1)} = j2\omega_n\sqrt{1-\zeta^2}$$

$$s_{1/2} = \frac{-2\omega_n\zeta \pm j2\omega_n\sqrt{1-\zeta^2}}{2}, \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\begin{cases} s_1 = -\omega_n(\zeta - j\sqrt{1-\zeta^2}) = -\omega_n\zeta - j\omega_d \\ s_2 = -\omega_n(\zeta + j\sqrt{1-\zeta^2}) = -\omega_n\zeta + j\omega_d \end{cases} \tag{25}$$



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