43. Diagonalizacja równania stanu

$$\underline{p} \in \mathbb{R}^n, \qquad \underline{p} \neq 0$$

$$\underline{A}\,\underline{p} = \underline{p}\lambda \quad \Rightarrow \quad (\lambda\underline{I} - \underline{A})\,\underline{p} = \underline{0}$$

$$\det(\lambda\underline{I} - \underline{A}) \neq 0$$

wartości własne $\lambda_1, \lambda_2, \ldots \lambda_n$: $\quad \det(\lambda\underline{I} - \underline{A}) = 0$
wektory własne $\underline{p}_1, \underline{p}_2, \ldots \underline{p}_n$: $\quad (\lambda_i\underline{I} - \underline{A})\underline{p}_i = \underline{0}, \qquad i = 1, 2, \ldots n$
przy czym $\lambda_i \neq \lambda_j$, $i, j = 1, 2, \ldots n$

$$\begin{cases} \underline{A}\,\underline{p}_1 = \underline{p}_1\lambda_1 \\ \underline{A}\,\underline{p}_2 = \underline{p}_2\lambda_2 \\ \;\ldots\ldots\ldots\ldots \\ \underline{A}\,\underline{p}_n = \underline{p}_n\lambda_n \end{cases} \tag{130}$$

$$\underline{A}\begin{bmatrix}\underline{p}_1 & \underline{p}_2 & \ldots & \underline{p}_n\end{bmatrix} = \begin{bmatrix}\underline{p}_1 & \underline{p}_2 & \ldots & \underline{p}_n\end{bmatrix}\begin{bmatrix}\lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda_n\end{bmatrix} \tag{131}$$

wprowadźmy oznaczenia:

$$\underline{P} = \begin{bmatrix}\underline{p}_1 & \underline{p}_2 & \ldots & \underline{p}_n\end{bmatrix} \in \mathbb{R}^{n\times n} \quad - \quad \text{macierz modalna}$$

$$\underline{\Lambda} = \begin{bmatrix}\lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda_n\end{bmatrix} \in \mathbb{R}^{n\times n} \quad - \quad \text{macierz wartości własnych}$$

układ równań (130) możemy zapisać:

$$\underline{A}\,\underline{P} = \underline{P}\,\underline{\Lambda} \qquad \Rightarrow \qquad \underline{\Lambda} = \underline{P}^{-1}\underline{A}\,\underline{P}$$

Wybierając przekształcenie liniowe:

$$\underline{x}(t) = \underline{T}\,\underline{x}^*(t) = \underline{P}\,\underline{x}^*(t) \tag{132}$$

otrzymujemy macierze $\underline{\widetilde{A}}$, $\underline{\widetilde{B}}$, $\underline{\widetilde{C}}$, $\underline{\widetilde{D}}$ o postaci:

$$\underline{\widetilde{A}} = \underline{P}^{-1}\underline{A}\,\underline{P} = \underline{\Lambda}, \qquad \underline{\widetilde{B}} = \underline{P}^{-1}\underline{B}, \qquad \underline{\widetilde{C}} = \underline{C}\underline{P}, \qquad \underline{\widetilde{D}} = \underline{D} \tag{133}$$

$$\begin{cases} \underline{\dot{x}}^* = \underline{\Lambda}\,\underline{x}^* + \underline{\widetilde{B}}\,\underline{u} \\ \underline{y} = \underline{\widetilde{C}}\,\underline{x}^* + \underline{\widetilde{D}}\,\underline{u} \end{cases} \tag{134}$$

44. Rozwiązanie równania stanu (o stałych współczynnikach)

Wykażemy, że rozwiązanie układu r.różn. niejednorodnych

$$\underline{\dot{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t) \tag{135}$$

spełniające warunek początkowy $\underline{x}(t_0) = \underline{x}_0$ ma postać:

$$\underline{x}(t) = e^{\underline{A}(t-t_0)}\underline{x}_0 + \int\limits_{t_0}^{t} e^{\underline{A}(t-\tau)}\underline{B}\,\underline{u}(\tau)d\tau. \tag{136}$$

Rozwiązanie układu r.r. jednorodnych $\underline{\dot{x}}(t) = \underline{A}\,\underline{x}(t)$ ma postać:

$$\underline{x}(t) = e^{\underline{A}t}$$

gdyż wtedy $\underline{\dot{x}}(t) = \underline{A}e^{\underline{A}t}$.

Poszukujemy rozw. układu r.różn. niejednorodnych w postaci:

$$\underline{x}(t) = e^{\underline{A}t}\underline{z}(t) \tag{137}$$

$$\dot{\underline{x}}(t) = \underline{A}e^{\underline{A}t}\underline{z}(t) + e^{\underline{A}t}\dot{\underline{z}}(t). \tag{138}$$

Wstawiamy do równania stanu (135):

$$\underline{\underline{A}e^{\underline{A}t}\underline{z}(t)} + e^{\underline{A}t}\dot{\underline{z}}(t) = \underline{\underline{A}e^{\underline{A}t}\underline{z}(t)} + \underline{B}\,\underline{u}(t)$$

$$\dot{\underline{z}}(t) = e^{-\underline{A}t}\underline{B}\,\underline{u}(t),$$

$$\underline{z}(t) = \underline{c}_1 + \int_{t_0}^{t} e^{-\underline{A}\tau}\underline{B}\,\underline{u}(\tau)d\tau. \tag{139}$$

Podstawiając (139) do (137) otrzymamy:

$$\underline{x}(t) = e^{\underline{A}t}\underline{c}_1 + \int_{t_0}^{t} e^{\underline{A}(t-\tau)}\underline{B}\,\underline{u}(\tau)d\tau$$

dla $t = t_0$ mamy $\underline{x}(t_0) = e^{\underline{A}t_0}\underline{c}_1 \quad \Rightarrow \quad \underline{c}_1 = e^{-\underline{A}t_0}\underline{x}(t_0)$ czyli

$$\underline{x}(t) = e^{\underline{A}(t-t_0)}\underline{x}_0 + \int_{t_0}^{t} e^{\underline{A}(t-\tau)}\underline{B}\,\underline{u}(\tau)d\tau.$$

$e^{\underline{A}t} = \underline{\Phi}(t)$ – macierz podstawowa (tranzycji). Obliczamy ją z szeregu McLaurina:

$$\underline{\Phi}(t) = e^{\underline{A}t} = \sum_{k=0}^{\infty}\frac{(\underline{A}t)^k}{k!} = \underline{I} + \underline{A}t + \frac{1}{2}\underline{A}^2t^2 + \frac{1}{6}\underline{A}^3t^3 + \ldots$$

$$\underline{x}(t) = \underline{\Phi}(t - t_0)\underline{x}_0 + \int\limits_{t_0}^{t} \underline{\Phi}(t - \tau)\underline{B}\,\underline{u}(\tau)d\tau. \qquad (140)$$

45. Wyznaczanie macierzy tranzycji

a) metoda odwrotnego przekształcenia Laplace'a

$$\dot{\underline{x}}(t) = \underline{A}\,\underline{x}(t), \quad \text{war. początkowy } \underline{x}(0) = \underline{x}_0 \qquad (141)$$

$$\underline{x}(t) = e^{\underline{A}t}\underline{x}_0 = \underline{\Phi}(t)\underline{x}_0 \qquad (142)$$

jest rozwiązaniem równania (141) ponieważ

$$\dot{\underline{x}}(t) = \underline{A}e^{\underline{A}t}\underline{x}_0 = \underline{A}\,\underline{\Phi}(t)\underline{x}_0$$

z drugiej strony do rozw. (141) stosujemy transf. Laplace'a:

$$s\underline{X}(s) - \underline{x}(0) = \underline{A}\,\underline{X}(s)$$
$$(s\underline{I} - \underline{A})\underline{X}(s) = \underline{x}(0)$$
$$\underline{X}(s) = (s\underline{I} - \underline{A})^{-1}\underline{x}(0)$$

$$\underline{x}(t) = \mathcal{L}^{-1}\left\{(s\underline{I} - \underline{A})^{-1}\right\}\underline{x}(0) \qquad (143)$$

porównując zależności (143) i (142) otrzymujemy:

$$\underline{\Phi}(t) = \mathcal{L}^{-1}\left\{(s\underline{I} - \underline{A})^{-1}\right\} \qquad (144)$$

Przykład

$$\underline{A} = \begin{bmatrix} -2 & -3 \\ 3 & 4 \end{bmatrix}$$

$$\det(s\underline{I} - \underline{A}) = (s+2)(s-4) + 9 = s^2 - 2s + 1 = (s-1)^2$$

$$(s\underline{I} - \underline{A})^{-1} = \begin{bmatrix} s+2 & 3 \\ -3 & s-4 \end{bmatrix}^{-1} = \frac{1}{(s-1)^2} \begin{bmatrix} s-4 & -3 \\ 3 & s+2 \end{bmatrix}$$

$$\mathcal{L}^{-1}\left\{\frac{s-4}{(s-1)^2}\right\} = (1-3t)e^t \mathbb{1}(t)$$

$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s-1)^2}\right\} = (1+3t)e^t \mathbb{1}(t)$$

$$\mathcal{L}^{-1}\left\{\frac{3}{(s-1)^2}\right\} = 3te^t \mathbb{1}(t)$$

zatem:

$$\underline{\Phi}(t) = \begin{bmatrix} (1-3t)e^t \mathbb{1}(t) & -3te^t \mathbb{1}(t) \\ 3te^t \mathbb{1}(t) & (1+3t)e^t \mathbb{1}(t) \end{bmatrix}$$

b) wykorzystanie wzoru Sylvestra

Zał. że równ. charakt. układu o macierzy stanu $\underline{A}$ ma postać:

$$\det(s\underline{I} - \underline{A}) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

ma jednokrotne pierwiastki (wartości własne) $\lambda_1, \lambda_2, \ldots \lambda_n$.
Wtedy:

$$\underline{\Phi}(t) = \sum_{k=1}^{n} \underline{Z}_k e^{\lambda_k t}, \quad \underline{Z}_k = \prod_{\substack{i=1 \\ i \neq k}}^{n} \frac{\underline{A} - \lambda_i \underline{I}}{\lambda_k - \lambda_i} \tag{145}$$

$$\underline{\Phi}(t) = \sum_{k=1}^{n} \underline{Z}_k e^{\lambda_k t}, \quad \underline{Z}_k = \prod_{\substack{i=1\\ i\neq k}}^{n} \frac{\underline{A} - \lambda_i \underline{I}}{\lambda_k - \lambda_i}$$

Przykład

$$\underline{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$\det(\lambda\underline{I} - \underline{A}) = \begin{vmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{vmatrix} = \lambda^2 + 3\lambda + 2, \quad \lambda_1 = -1,\ \lambda_2 = -2$$

$$\underline{Z}_1 = \frac{\underline{A} - \lambda_2\underline{I}}{\lambda_1 - \lambda_2} = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}, \quad \underline{Z}_2 = \frac{\underline{A} - \lambda_1\underline{I}}{\lambda_2 - \lambda_1} = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$$

$$\underline{\Phi}(t) = \left(\underline{Z}_1 e^{\lambda_1 t} + \underline{Z}_2 e^{\lambda_2 t}\right) \mathbb{1}(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \mathbb{1}(t)$$

46. Własności macierzy tranzycji

a) $\underline{\Phi}(0) = e^{\underline{A}0} = \underline{I}$

b) $\underline{\Phi}(t) = e^{\underline{A}t} = \left(e^{-\underline{A}t}\right)^{-1} = \left(\underline{\Phi}(-t)\right)^{-1}$ lub $\underline{\Phi}^{-1}(t) = \underline{\Phi}(-t)$

c) $\underline{\Phi}(t_1 + t_2) = e^{\underline{A}(t_1+t_2)} = e^{\underline{A}t_1} e^{\underline{A}t_2} = \underline{\Phi}(t_1)\underline{\Phi}(t_2) = \underline{\Phi}(t_2)\underline{\Phi}(t_1)$

d) $\left(\underline{\Phi}(t)\right)^n = \underline{\Phi}(nt)$

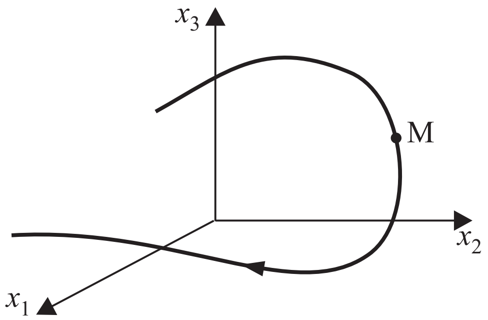
e) $\underline{\Phi}(t_2 - t_1)\underline{\Phi}(t_1 - t_0) = \underline{\Phi}(t_2 - t_0) = \underline{\Phi}(t_1 - t_0)\underline{\Phi}(t_2 - t_1)$

f) $\underline{A} = \underline{\Lambda} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \Rightarrow \underline{\Phi}(t) = e^{\underline{A}t} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$

47. Zmienne stanu fazowe

współrzędne fazowe: $x_2 = \dfrac{dx_1}{dt} = \dot{x}_1,\ \ x_3 = \dot{x}_2,\ \ x_n = \dot{x}_{n-1}$

wektor fazowy: $\underline{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$



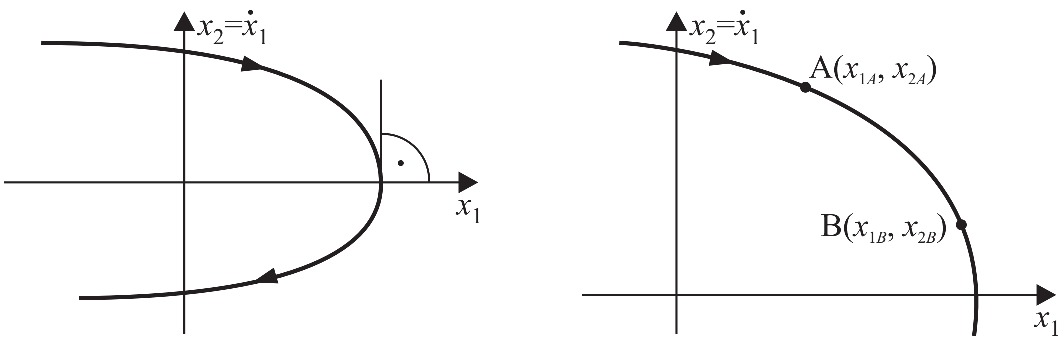
Rys. 133.

<u>Własności trajektorii fazowej</u>

a) $x_2 = \dfrac{dx_1}{dt}$; wtedy $x_2 > 0 \quad \Rightarrow \quad dx_1 > 0$

b) $x_2 < 0 \quad \Rightarrow \quad dx_1 < 0$

c) $x_2 = 0 \quad \Rightarrow \quad dx_1 = 0$

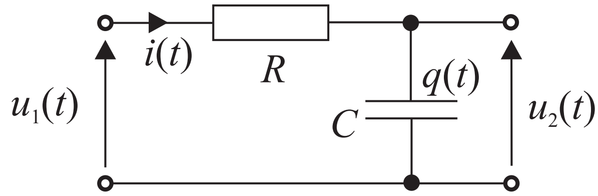


(a) Rys. 134. (b)

$$x_2 = \frac{dx_1}{dt} \quad \Rightarrow \quad dt = \frac{dx_1}{x_2} \quad \Rightarrow \quad \int_{t_A}^{t_B} dt = \int_{x_{1A}}^{x_{1B}} \frac{dx_1}{x_2}$$

$$\Delta t = t_B - t_A = \int\limits_{x_{1A}}^{x_{1B}} \frac{dx_1}{x_2} \qquad (146)$$

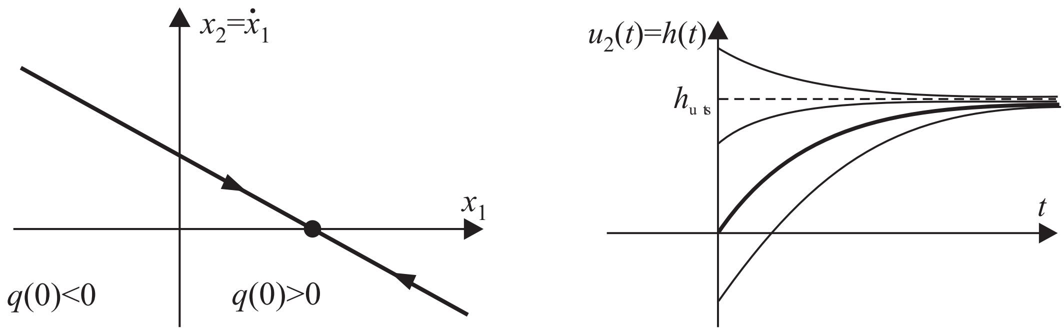
Przykład



Rys. 135.

$$x_1(t) = q(t), \qquad x_2(t) = i(t) = \frac{dq(t)}{dt} = \dot{x}_1(t), \qquad u(t) = u_1(t)$$

$$u_1(t) = Ri(t) + \frac{q(t)}{C} \;\Rightarrow\; u(t) = Rx_2(t) + \frac{x_1(t)}{C} \;\Rightarrow\; x_2 = -\frac{1}{RC}x_1 + \frac{1}{R}u$$



(a) (b)

Rys. 136.

48. Wykorzystanie współrzędnych fazowych do wyznaczania równania stanu

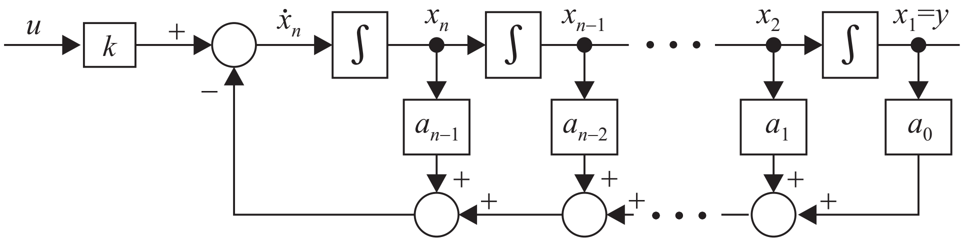
$$G(s) = \frac{Y(s)}{U(s)} = \frac{k}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \tag{147}$$

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1\dot{y}(t) + a_0 y(t) = ku(t) \tag{148}$$

$$x_1(t) = y(t), \quad x_2(t) = \dot{y}(t), \quad \ldots \quad x_n(t) = y^{(n-1)}(t)$$

$$\dot{x}_n(t) + a_{n-1}x_n(t) + \cdots + a_1 x_2(t) + a_0 x_1(t) = ku(t)$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_0 x_1 - a_1 x_2 - \dots - a_{n-1} x_n + ku \end{cases} \tag{149}$$



Rys. 137.

$$\begin{cases} \underline{\dot{x}} = \underline{A}_f \underline{x} + \underline{b}_f \underline{u} \\ \underline{y} = \underline{c}_f \underline{x} + d_f \underline{u} \end{cases} \qquad \underline{c}_f = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad d_f = 0$$

$$\underline{A}_f = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix}, \quad \underline{b}_f = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ k \end{bmatrix}$$

49. Wybór zmiennych stanu dla układu o znanej transmitancji

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}, \quad m < n \tag{150}$$

Metoda bezpośrednia

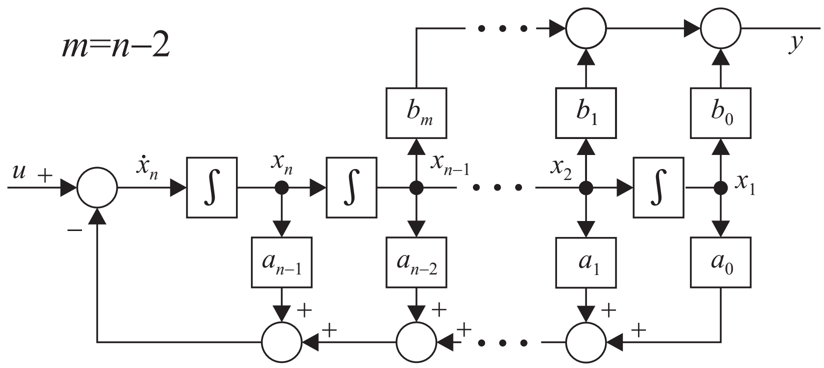
$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_m s^{m-n}+b_{m-1}s^{m-1-n}+\cdots+b_1 s^{1-n}+b_0 s^{-n}}{1+a_{n-1}s^{-1}+\cdots+a_1 s^{1-n}+a_0 s^{-n}}$$

$$Y(s)=(b_m s^{m-n}+b_{m-1}s^{m-1-n}+\cdots+b_1 s^{1-n}+b_0 s^{-n})X(s) \quad (151)$$

$$X(s)=\frac{U(s)}{1+a_{n-1}s^{-1}+\cdots+a_1 s^{1-n}+a_0 s^{-n}}$$

$$X(s)=U(s)-(a_{n-1}s^{-1}+\cdots+a_1 s^{1-n}+a_0 s^{-n})X(s) \quad (152)$$

$$\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=x_3\\ \ldots\\ \dot{x}_{n-1}=x_n\\ \dot{x}_n=-a_0x_1-a_1x_2-\cdots-a_{n-1}x_n+u\\ y=b_0x_1+b_1x_2+\cdots+b_mx_{m+1}\end{cases} \quad (153)$$



Rys. 138.

$$\begin{cases}\underline{\dot{x}}=\underline{A}\,\underline{x}+\underline{b}\,\underline{u}\\ \underline{y}=\underline{c}\,\underline{x}+d\underline{u}\end{cases} \qquad \underline{c}=\begin{bmatrix}b_0 & b_1 & \ldots & b_m & 0 & \ldots & 0\end{bmatrix}, \quad d=0$$

$$\underline{A}=\begin{bmatrix}0 & 1 & 0 & \ldots & 0 & 0\\ 0 & 0 & 1 & \ldots & 0 & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & 0 & \ldots & 0 & 1\\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-2} & -a_{n-1}\end{bmatrix}, \quad \underline{b}=\begin{bmatrix}0\\0\\ \vdots\\0\\1\end{bmatrix}$$

Metoda równoległa

$$M(s) = s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{L(s)}{M(s)} = \sum_{k=1}^{n} \frac{h_k}{s - s_k}, \quad h_k = \frac{L(s_k)}{M'(s_k)} \tag{154}$$

$$M'(s) = \frac{dM(s)}{ds}, \qquad L(s) = b_m s^m + b_{m-1}s^{m-1} + \dots + b_1 s + b_0$$

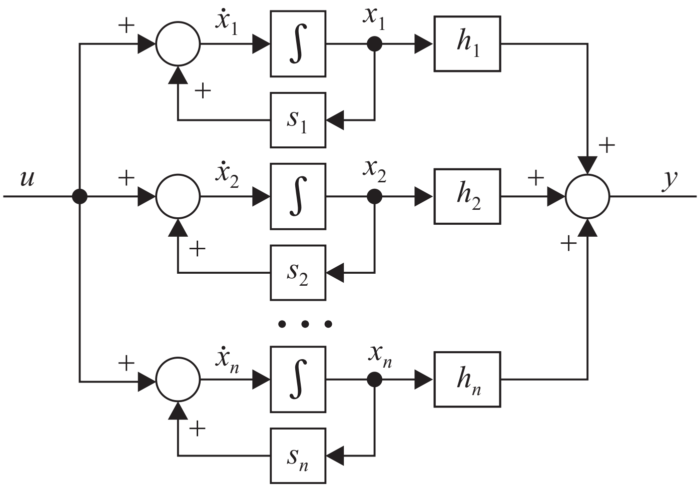
$$Y(s) = \sum_{k=1}^{n} \frac{h_k}{s - s_k} U(s) = \sum_{k=1}^{n} h_k X_k(s), \quad X_k(s) = \frac{U(s)}{s - s_k} \tag{155}$$

$$sX_k(s) - s_k X_k(s) = U(s) \quad \Rightarrow \quad \dot{x}_k = s_k x_k + u$$

$$\begin{cases} \dot{x}_1 = s_1 x_1 + u \\ \dot{x}_2 = s_2 x_2 + u \\ \dots \\ \dot{x}_n = s_n x_n + u \\ y = h_1 x_1 + h_2 x_2 + \dots + h_n x_n \end{cases} \qquad \begin{cases} \underline{\dot{x}} = \underline{A}\,\underline{x} + \underline{b}\,\underline{u} \\ \underline{y} = \underline{c}\,\underline{x} + d\underline{u} \end{cases} \tag{156}$$

$$\underline{A} = \begin{bmatrix} s_1 & 0 & \dots & 0 \\ 0 & s_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & s_n \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \underline{c} = \begin{bmatrix} h_1 & h_2 & \dots & h_n \end{bmatrix}$$

$$d = 0$$



Rys. 139.