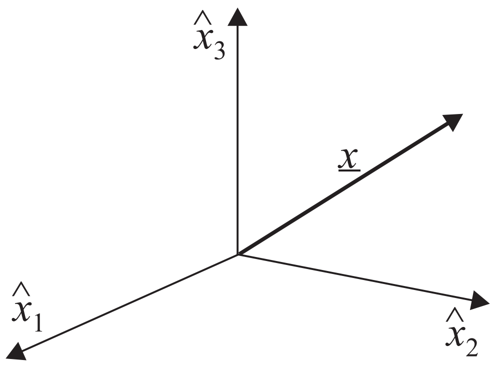
38. Wektor stanu i przestrzeń stanu

$$t \geqslant t_0$$

$$\underline{x} \qquad x_1, x_2, \ldots x_n \qquad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}^T$$



Rys. 129.

39. Równanie stanu i równanie wyjścia

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{p1} & c_{p2} & \dots & c_{pn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \dots & d_{1m} \\ d_{21} & d_{22} & \dots & d_{2m} \\ \dots & \dots & \dots & \dots \\ d_{p1} & d_{p2} & \dots & d_{pm} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}$$

$$\dot{x}_i = \frac{dx_i}{dt}$$

$$\underline{\dot{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t) \quad \text{– równanie stanu} \tag{120}$$

$$\underline{y}(t) = \underline{C}\,\underline{x}(t) + \underline{D}\,\underline{u}(t) \quad \text{– równanie wyjścia} \tag{121}$$

$\underline{x}(t) \in \mathbb{R}^n$ – wektor stanu układu
$\underline{u}(t) \in \mathbb{R}^m$ – wektor sygnałów wejściowych
$\underline{y}(t) \in \mathbb{R}^p$ – wektor sygnałów wyjściowych
$\underline{A} \in \mathbb{R}^{n\times n}$ – macierz stanu układu
$\underline{B} \in \mathbb{R}^{n\times m}$ – macierz wejściowa
$\underline{C} \in \mathbb{R}^{p\times n}$ – macierz wyjściowa
$\underline{D} \in \mathbb{R}^{p\times m}$ – macierz bezpośredniego połączenia wejścia z wyjściem

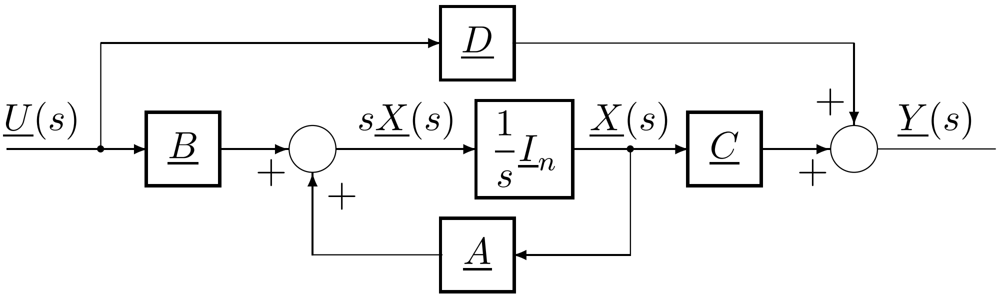
dla układu niestacjonarnego:

$$\underline{\dot{x}}(t) = \underline{A}(t)\,\underline{x}(t) + \underline{B}(t)\,\underline{u}(t)$$
$$\underline{y}(t) = \underline{C}(t)\,\underline{x}(t) + \underline{D}(t)\,\underline{u}(t)$$

w postaci operatorowej:

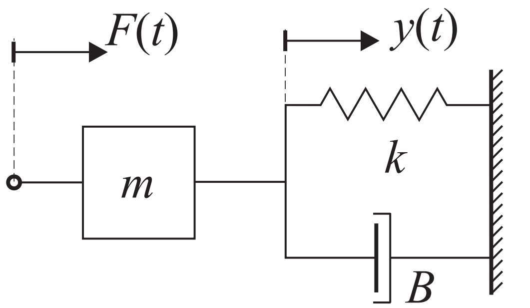
$$s\underline{X}(s) = \underline{A}\,\underline{X}(s) + \underline{B}\,\underline{U}(s) \tag{122}$$

$$\underline{Y}(s) = \underline{C}\,\underline{X}(s) + \underline{D}\,\underline{U}(s) \tag{123}$$



Rys. 130.

Przykład



Rys. 131.

$$F(t) = m\frac{d^2y(t)}{dt^2} + B\frac{dy(t)}{dt} + ky(t) = m\ddot{y}(t) + B\dot{y}(t) + ky(t)$$

wybieramy $x_1(t) = y(t)$, $\quad x_2(t) = \dot{y}(t) = \dot{x}_1(t)$, $\quad u(t) = F(t)$

$$u = m\dot{x}_2 + Bx_2 + kx_1 \qquad y = x_1$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{B}{m}x_2 + \frac{1}{m}u \qquad \underline{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$$

$$\dot{x}_1 = x_2$$

$$\underline{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{B}{m} \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \quad \underline{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \underline{D} = \begin{bmatrix} 0 \end{bmatrix}$$

40. Transmitancja wielowymiarowa

- układy o jednym wejściu i jednym wyjściu (ang. *single-input, single-output* – SISO)
- układy o wielu wejściach i wielu wyjściach (ang. *multiple-input, multiple-output* – MIMO)

(a)

Rys. 132.

(b)

$$\underline{G}(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \dots & G_{1m}(s) \\ G_{21}(s) & G_{22}(s) & \dots & G_{2m}(s) \\ \dots & \dots & \dots & \dots \\ G_{p1}(s) & G_{p2}(s) & \dots & G_{pm}(s) \end{bmatrix} \in \mathbb{R}^{p\times m} \qquad (124)$$

$$\underline{Y}(s) = \underline{G}(s)\underline{U}(s) \qquad (125)$$

przy czym $\underline{U}(s) = \mathcal{L}\{\underline{u}(t)\}, \quad \underline{Y}(s) = \mathcal{L}\{\underline{y}(t)\}$

$$G_{ij}(s) \triangleq \frac{Y_i(s)}{U_j(s)}, \quad i = 1, 2, \dots p, \quad j = 1, 2, \dots m \qquad (126)$$

$$Y_i(s) = \sum_{j=1}^{m} G_{ij}(s)U_j(s), \qquad y_i(t) = \mathcal{L}^{-1}\{Y_i(s)\}$$

41. Związek między opisem w przestrzeni stanu a transmitancją

$$\begin{cases} \underline{\dot{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t) \\ \underline{y}(t) = \underline{C}\,\underline{x}(t) + \underline{D}\,\underline{u}(t) \end{cases} \Rightarrow \begin{cases} s\underline{X}(s) = \underline{A}\,\underline{X}(s) + \underline{B}\,\underline{U}(s) \\ \underline{Y}(s) = \underline{C}\,\underline{X}(s) + \underline{D}\,\underline{U}(s) \end{cases}$$

$$(s\underline{I} - \underline{A})\underline{X}(s) = \underline{B}\,\underline{U}(s)$$

$$\underline{X}(s) = (s\underline{I} - \underline{A})^{-1}\underline{B}\,\underline{U}(s)$$

wstawiamy do równania wyjścia:

$$\underline{Y}(s) = \underline{C}(s\underline{I} - \underline{A})^{-1}\underline{B}\,\underline{U}(s) + \underline{DU}(s),$$

a wielowymiarowa transmitancja operatorowa ma postać:

$$\underline{G}(s) = \underline{C}(s\underline{I} - \underline{A})^{-1}\underline{B} + \underline{D} \tag{127}$$

ponieważ $\underline{Y}(s) = \underline{G}(s)\underline{U}(s)$

Przykład

$$\begin{cases} \underline{\dot{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u} \\ \underline{y} = \underline{C}\,\underline{x} + \underline{D}\,\underline{u} \end{cases} \qquad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\underline{A} = \begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \underline{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \underline{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\underline{G}(s) = ?$

$$s\underline{I} - \underline{A} = \begin{bmatrix} s+1 & 1 \\ -3 & s \end{bmatrix}, \quad \det(s\underline{I} - \underline{A}) = (s+1)s + 3 = s^2 + s + 3$$

Macierz odwrotna – przypomnienie

zał. $\underline{M} = [m_{ij}]$ kwadratowa i nieosobliwa, tzn. $\det(\underline{M}) \neq 0$

$$\underline{M}^{-1} = \frac{\underline{M}^D}{\det(\underline{M})} \quad \text{– macierz odwrotna}$$

$$\underline{M}^D = \left[(-1)^{i+j} M_{ij}\right]^T \quad \text{– macierz dołączona}$$

$$M_{ij} \quad \text{– minor wyznacznika związany z } m_{ij}$$

$$(s\underline{I} - \underline{A})^{-1} = \frac{1}{s^2+s+3}\begin{bmatrix} s & -1 \\ 3 & s+1 \end{bmatrix}$$

$$\underline{G}(s) = \underline{C}(s\underline{I} - \underline{A})^{-1}\underline{B} = \frac{1}{s^2+s+3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s & -1 \\ 3 & s+1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} =$$

$$= \frac{1}{s^2+s+3}\begin{bmatrix} s-1 & s \\ s+4 & 3 \end{bmatrix}$$

Stąd:

$$\underline{G}(s) = \begin{bmatrix} \dfrac{s-1}{s^2+s+3} & \dfrac{s}{s^2+s+3} \\ \dfrac{s+4}{s^2+s+3} & \dfrac{3}{s^2+s+3} \end{bmatrix}$$

$$\underline{G}(s) = \underline{C}(s\underline{I} - \underline{A})^{-1}\underline{B} + \underline{D} = \underline{C}\frac{(s\underline{I} - \underline{A})^D}{\det(s\underline{I} - \underline{A})}\underline{B} + \underline{D}$$

stąd równanie charakterystyczne: $\det(s\underline{I} - \underline{A}) = 0$

(gdy $G(s) = \dfrac{L(s)}{M(s)} \rightarrow$ równanie charakterystyczne $M(s) = 0$)

42. Przekształcenie nieosobliwe zmiennych stanu

$$\underline{x}(t) = \underline{T}\,\underline{x}^*(t) \tag{128}$$

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u}$$
$$\underline{T}\,\dot{\underline{x}}^* = \underline{A}\,\underline{T}\,\underline{x}^* + \underline{B}\,\underline{u}$$

$$\begin{cases} \dot{\underline{x}}^* = \underline{T}^{-1}\underline{A}\,\underline{T}\,\underline{x}^* + \underline{T}^{-1}\underline{B}\,\underline{u} \\ \underline{y} = \underline{CT}\,\underline{x}^* + \underline{D}\,\underline{u} \end{cases} \quad \Rightarrow \begin{cases} \dot{\underline{x}}^* = \underline{\widetilde{A}}\,\underline{x}^* + \underline{\widetilde{B}}\,\underline{u} \\ \underline{y} = \underline{\widetilde{C}}\,\underline{x}^* + \underline{\widetilde{D}}\,\underline{u} \end{cases}$$

przy czym

$$\underline{\widetilde{A}} = \underline{T}^{-1}\underline{A}\,\underline{T}, \qquad \underline{\widetilde{B}} = \underline{T}^{-1}\underline{B}, \qquad \underline{\widetilde{C}} = \underline{CT}, \qquad \underline{\widetilde{D}} = \underline{D} \tag{129}$$

$$\det(s\underline{I} - \underline{A}) = 0$$
$$\det(s\underline{I} - \underline{\widetilde{A}}) = 0$$

$$\det(s\underline{I} - \underline{\widetilde{A}}) = \det(s\underline{I} - \underline{T}^{-1}\underline{A}\,\underline{T}) = \det(s\underline{T}^{-1}\underline{T} - \underline{T}^{-1}\underline{A}\,\underline{T}) =$$
$$= \det\left[\underline{T}^{-1}(s\underline{I})\underline{T} - \underline{T}^{-1}\underline{A}\,\underline{T}\right] = \det\left[\underline{T}^{-1}(s\underline{I} - \underline{A})\underline{T}\right] =$$
$$= \det(\underline{T}^{-1})\det(s\underline{I} - \underline{A})\det(\underline{T}) = \det(s\underline{I} - \underline{A})$$

ponieważ $\quad \underline{T}^{-1}\underline{T} = \underline{I} \quad \Rightarrow \quad \det(\underline{T}^{-1})\det(\underline{T}) = 1$