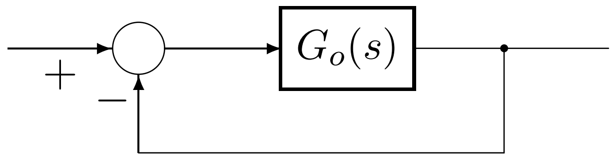
Przykład $$G_o(s) = \frac{k}{s(Ts+1)^2}, \qquad T = 0{,}1[\mathrm{s}]$$



Rys. 38. $\quad \Delta M \geqslant 8[\mathrm{dB}], \quad \Delta\varphi \geqslant 30^\circ = \pi/6$

$$|G_o(j\omega)| = \frac{k}{\omega(\omega^2T^2+1)}, \qquad \varphi_o(\omega) = -\frac{\pi}{2} - 2\mathrm{arctg}\omega T$$

Zapas modułu:

$$\begin{cases} \arg G_o(j\omega_{-\pi}) = -\pi \\ -20\lg|G_o(j\omega_{-\pi})| \geqslant \Delta M = 8[\mathrm{dB}] \quad (20\lg|G_o(j\omega_{-\pi})| \leqslant -\Delta M) \end{cases}$$

$$-\frac{\pi}{2} - 2\mathrm{arctg}\omega T = -\pi \;\rightarrow\; -\mathrm{arctg}\omega T = -\frac{\pi}{4} \;\rightarrow\; \omega_{-\pi} = \frac{1}{T} = 10[\mathrm{rad/s}]$$

$$20\lg|G_o(j\omega_{-\pi})| \leqslant -8 \;\rightarrow\; \lg|G_o(j\omega_{-\pi})| \leqslant -0{,}4$$

$$|G_o(j\omega_{-\pi})| \leqslant 10^{-0{,}4} = 0{,}398 \approx 0{,}4$$

$$|G_o(j\omega_{-\pi})| = \frac{k}{\omega_{-\pi}(\omega_{-\pi}^2T^2+1)} = \frac{k}{10(1+1)} = \frac{k}{20} \leqslant 0{,}4 \;\rightarrow\; k \leqslant 8$$

Zapas fazy:

$$\begin{cases} \arg G_o(j\omega_p) = -\pi + \Delta\varphi \\ 20\lg|G_o(j\omega_p)| \leqslant 0 \qquad (|G_o(j\omega_p)| \leqslant 1) \end{cases}$$

$$-\pi/2 - 2\mathrm{arctg}\omega T = -\pi + \pi/6 = -5\pi/6 \;\rightarrow\; -2\mathrm{arctg}\omega T = -\pi/3$$

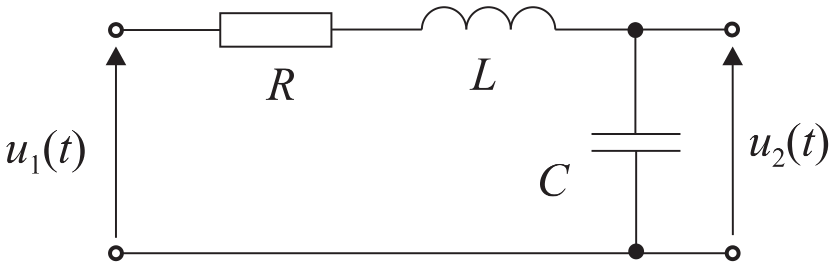
$$\mathrm{arctg}\omega T = \frac{\pi}{6} \;\rightarrow\; \omega_p T = \sqrt{3}/3 \;\rightarrow\; \omega_p = 10\sqrt{3}/3$$

$$|G_o(j\omega_p)| = \frac{k}{\omega_p(\omega_p^2T^2+1)} = \frac{k}{10\sqrt{3}/3(1/3+1)} = \frac{9k}{40\sqrt{3}} \leqslant 1$$

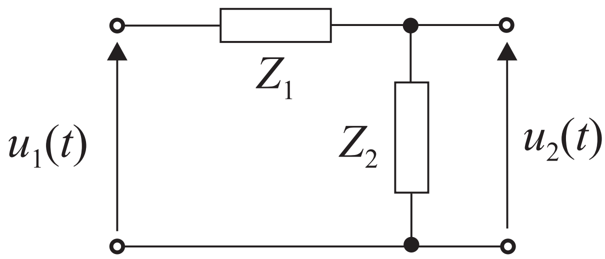
$$k \leqslant 40\sqrt{3}/9 \approx 7{,}69$$

14. Układy dynamiczne drugiego rzędu

Przykład



Rys. 39.



Rys. 40.

$$Z_1(s) = R + sL \qquad\qquad Z_2(s) = \frac{1}{sC}$$

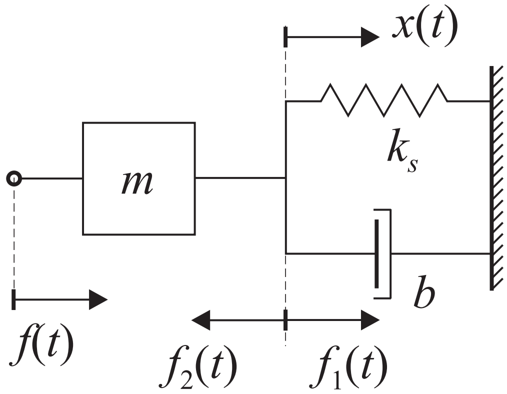
$$G(s) = \frac{U_2(s)}{U_1(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} = \frac{\frac{1}{sC}}{R + sL + \frac{1}{sC}} =$$

$$= \frac{1}{s^2LC + sRC + 1} = \frac{\frac{1}{LC}}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$

$$G(s) = \frac{k}{T_n^2 s^2 + 2\zeta T_n s + 1}$$

$$k = 1, \quad T_n = \sqrt{LC}, \quad 2\zeta T_n = RC \rightarrow \zeta = \frac{RC}{2T_n} = \frac{RC}{2\sqrt{LC}} = \frac{R}{2}\sqrt{\frac{C}{L}}$$

Przykład



Rys. 41.

$$f(t) = m\frac{d^2x(t)}{dt^2} + f_1(t),$$

$$f_2(t) = k_s\left[0 - x(t)\right] + b\left[0 - \frac{dx(t)}{dt}\right],$$

$$f_1(t) + f_2(t) = 0,$$

$$f(t) = m\frac{d^2x(t)}{dt^2} - f_2(t) = m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + k_s x(t),$$

$$F(s) = (ms^2 + bs + k_s)X(s),$$

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k_s} = \frac{\frac{1}{k_s}}{\frac{m}{k_s}s^2 + \frac{b}{k_s}s + 1}$$

$$G(s) = \frac{k}{T_n^2 s^2 + 2\zeta T_n s + 1}$$

$$k = \frac{1}{k_s}, \quad T_n = \sqrt{\frac{m}{k_s}}$$

$$2\zeta T_n = \frac{b}{k_s} \rightarrow \zeta = \frac{b}{2k_s T_n} = \frac{b}{2k_s}\sqrt{\frac{k_s}{m}} = \frac{b}{2\sqrt{k_s m}}$$

$$G(s) = \frac{k}{T_n^2 s^2 + 2\zeta T_n s + 1} = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{39}$$

k – wzmocnienie,

T_n – stała czasowa,

$\omega_n = 1/T_n > 0$ – pulsacja drgań nietłumionych (p. naturalna),

ζ – współczynnik tłumienia (rozważamy $\zeta \geqslant 0$),

równanie charakterystyczne:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\Delta = 4\omega_n^2(\zeta^2 - 1)$$

a) <u>układ przetłumiony</u> (ang. overdamped): $\zeta > 1$, $\Delta > 0$

$$\sqrt{\Delta} = 2\omega_n\sqrt{\zeta^2 - 1}$$

$$s_{1/2} = \frac{-2\omega_n\zeta \pm 2\omega_n\sqrt{\zeta^2 - 1}}{2}$$

$$s_1 = -\omega_n(\zeta - \sqrt{\zeta^2 - 1}), \qquad s_2 = -\omega_n(\zeta + \sqrt{\zeta^2 - 1}), \tag{40}$$

$$s_1 s_2 = \omega_n^2\left(\zeta^2 - (\zeta^2 - 1)\right) = \omega_n^2$$

$$G(s) = \frac{k\omega_n^2}{(s - s_1)(s - s_2)} = \frac{k\omega_n^2}{s_1 s_2\left(-\frac{1}{s_1}s + 1\right)\left(-\frac{1}{s_2}s + 1\right)} =$$

$$= \frac{k}{(T_1 s + 1)(T_2 s + 1)}, \tag{41}$$

gdzie $T_1 = -\frac{1}{s_1}$, $T_2 = -\frac{1}{s_2}$, a wtedy $T_1 T_2 = \frac{1}{\omega_n^2}$, $T_1 + T_2 = \frac{2\zeta}{\omega_n}$

$$h(t) = k\left(1 - \frac{T_1 e^{-t/T_1} - T_2 e^{-t/T_2}}{T_1 - T_2}\right)\mathbb{1}(t)$$

b) układ tłumiony krytycznie: $\zeta = 1$, $\Delta = 0$

$$s_0 = -\frac{2\zeta\omega_n}{2} = -\zeta\omega_n = -\omega_n, \tag{42}$$

$$G(s) = \frac{k\omega_n^2}{(s-s_0)^2} = \frac{k\omega_n^2}{(s+\omega_n)^2} = \frac{k}{(Ts+1)^2}, \tag{43}$$

gdzie $T = -\frac{1}{s_0} = \frac{1}{\omega_n}$

$$h(t) = \mathcal{L}^{-1}\left\{G(s)\frac{1}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{k\omega_n^2}{(s+\omega_n)^2 s}\right\} = k\left[1-(1+\omega_n t)e^{-\omega_n t}\right]\mathbb{1}(t)$$

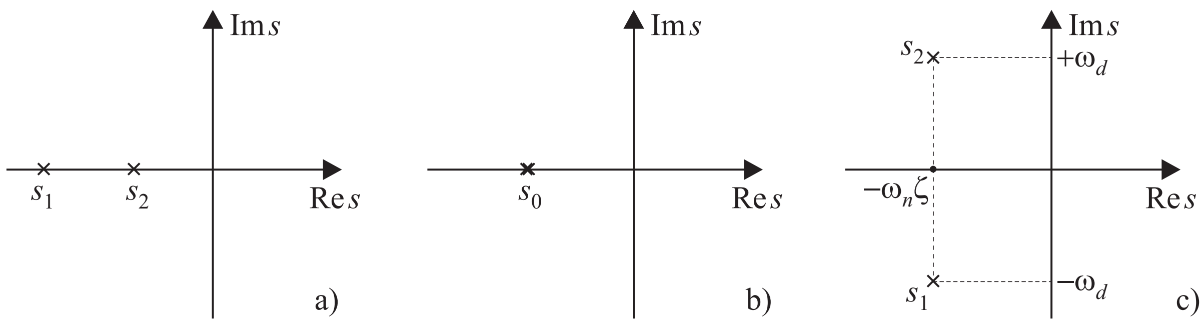
przebieg aperiodyczny krytyczny (44)

c) układ niedotłumiony (ang. underdamped): $0 \leqslant \zeta < 1$, $\Delta < 0$

$$\sqrt{\Delta} = \sqrt{4\omega_n^2(\zeta^2 - 1)} = j2\omega_n\sqrt{1-\zeta^2}$$

$$s_{1/2} = \frac{-2\omega_n\zeta \pm j2\omega_n\sqrt{1-\zeta^2}}{2}, \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\begin{cases} s_1 = -\omega_n(\zeta - j\sqrt{1-\zeta^2}) = -\omega_n\zeta - j\omega_d \\ s_2 = -\omega_n(\zeta + j\sqrt{1-\zeta^2}) = -\omega_n\zeta + j\omega_d \end{cases} \tag{45}$$



Rys. 42.

15. Odpowiedź skokowa elementu oscylacyjnego

$$G(s) = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{k\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)} = \\ = \frac{k\omega_n^2}{(s + \zeta\omega_n)^2 + \omega_d^2} \tag{46}$$

przy czym $\omega_d = \omega_n\sqrt{1-\zeta^2}$ – tłumiona pulsacja naturalna

$$H(s) = \frac{G(s)}{s} = \frac{k\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)s} = k\frac{s^2 + 2\zeta\omega_n s + \omega_n^2 - s^2 - 2\zeta\omega_n s}{(s^2 + 2\zeta\omega_n s + \omega_n^2)s} = \\ = k\left(\frac{1}{s} - \frac{s^2 + 2\zeta\omega_n s}{(s^2 + 2\zeta\omega_n s + \omega_n^2)s}\right) = k\left(\frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) = \\ = k\left(\frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}\right) = \\ = k\left(\frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right) \tag{47}$$

$$\left\{\begin{array}{l} \mathcal{L}^{-1}\left[\dfrac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] = e^{-\zeta\omega_n t}\cos(\omega_d t)\mathbb{1}(t) \\ \mathcal{L}^{-1}\left[\dfrac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] = e^{-\zeta\omega_n t}\sin(\omega_d t)\mathbb{1}(t) \end{array}\right\} \tag{48}$$

$$h(t) = \mathcal{L}^{-1}\left\{H(s)\right\} = \mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\} = \\ = k\left(1 - e^{-\zeta\omega_n t}\cos(\omega_d t) - \frac{\zeta\omega_n}{\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t)\right)\mathbb{1}(t) = \\ = k\left[1 - e^{-\zeta\omega_n t}\left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right)\right]\mathbb{1}(t)$$

$$h(t) = k\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sqrt{1-\zeta^2}\cos(\omega_d t) + \zeta\sin(\omega_d t)\right)\right]\mathbb{1}(t) =$$

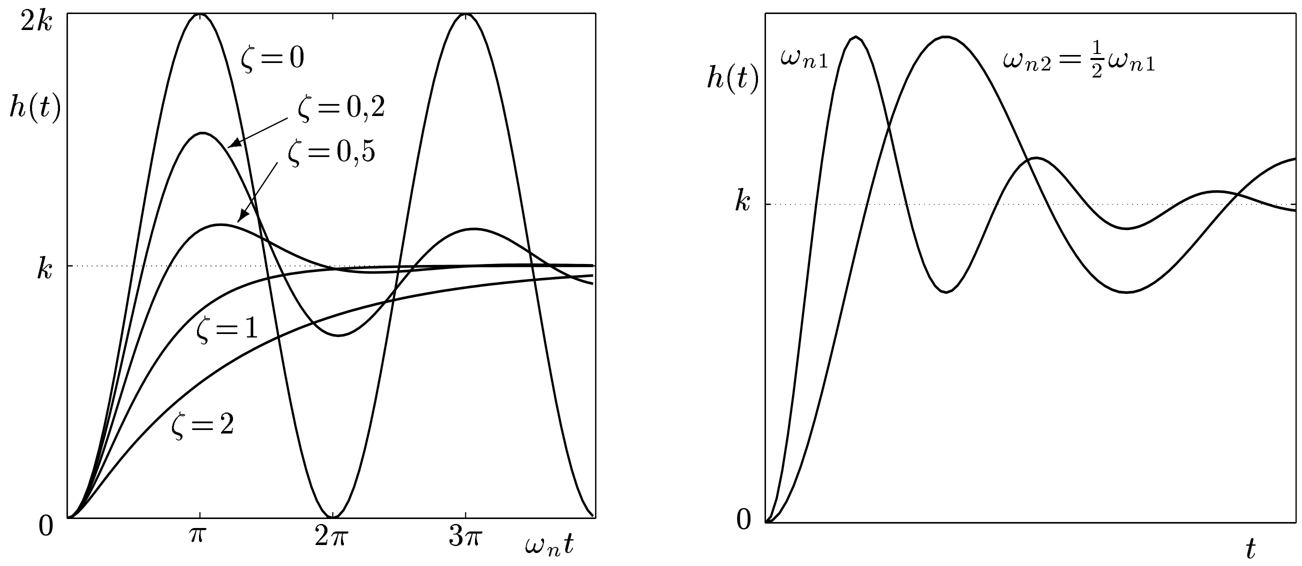
$$= \left\{\begin{array}{c}\zeta = \cos\theta\\ \sqrt{1-\zeta^2} = \sin\theta\end{array}\right\} =$$

$$= k\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\big(\sin\theta\cos(\omega_d t) + \cos\theta\sin(\omega_d t)\big)\right]\mathbb{1}(t) =$$

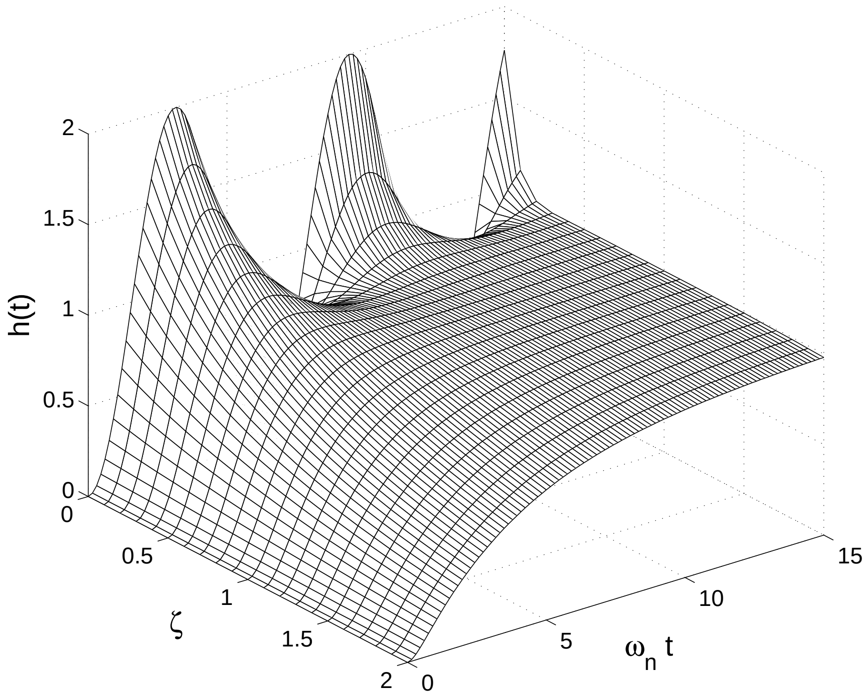
$$= k\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)\right]\mathbb{1}(t)$$

przy czym $\theta = \mathrm{arctg}\dfrac{\sin\theta}{\cos\theta} = \mathrm{arctg}\dfrac{\sqrt{1-\zeta^2}}{\zeta}$, $\quad \omega_d = \omega_n\sqrt{1-\zeta^2}$

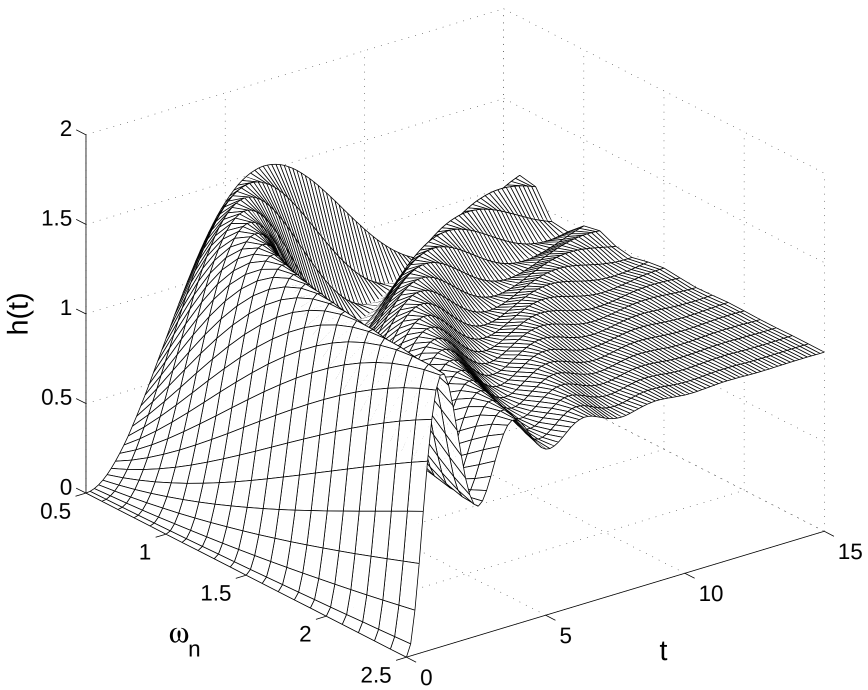
$$h(t)\Big|_{\zeta=0} = k\left[1 - \sin\left(\omega_n t + \frac{\pi}{2}\right)\right]\mathbb{1}(t) = k\left[1 - \cos(\omega_n t)\right]\mathbb{1}(t) \qquad (49)$$



(a) Rys. 43. (b)



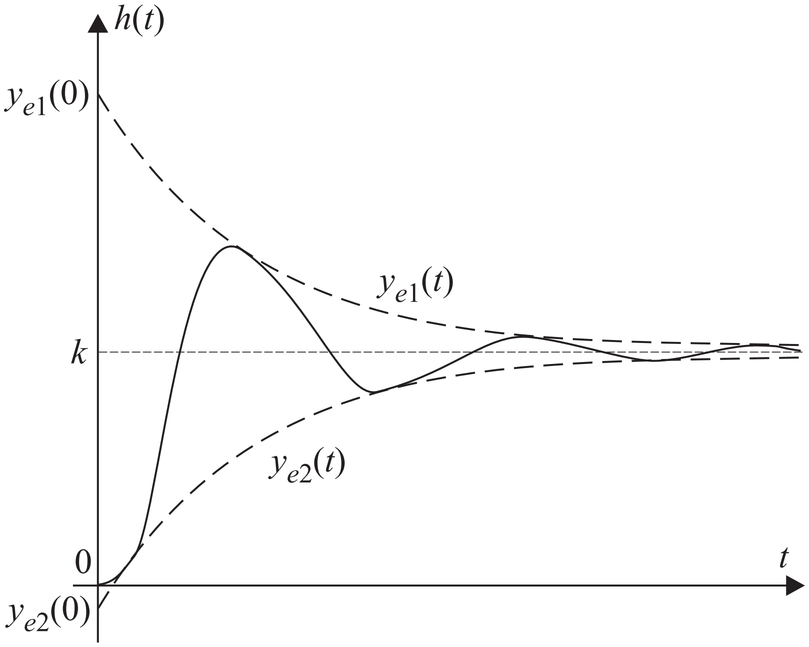
Rys. 44. $k = 1$



Rys. 45. $k = 1$, $\zeta = 0{,}2$

przyjmując $\sin(\cdot) = \pm 1$ otrzymujemy z $h(t)$ równanie obwiedni:

$$y_{e1/2}(t) = k\left(1 \pm \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\mathbb{1}(t) \tag{50}$$



Rys. 46.

$$y_{e1/2}(0) = k\left(1 \pm \frac{1}{\sqrt{1-\zeta^2}}\right), \qquad T_e = \frac{1}{\zeta\omega_n}$$

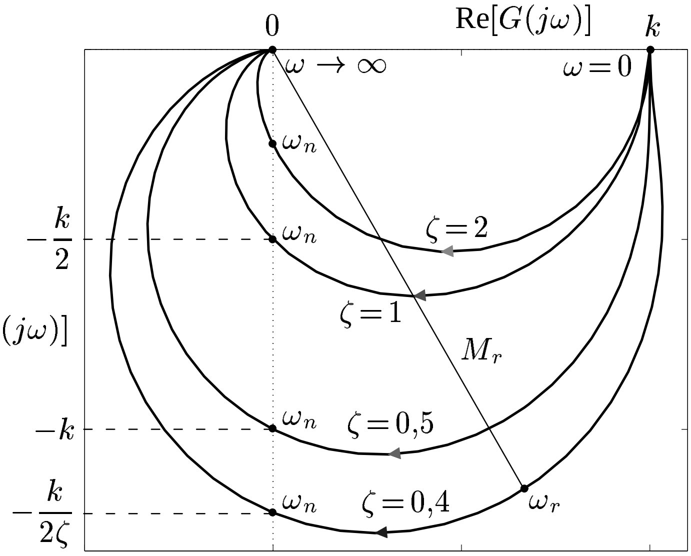
16. Charakterystyka amplitudowo-fazowa elementu oscylacyjnego

$$G(j\omega) = \frac{k\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} = \frac{k\omega_n^2}{(\omega_n^2 - \omega^2) + j\cdot 2\zeta\omega_n\omega} =$$
$$= k\omega_n^2 \frac{(\omega_n^2-\omega^2) - j\cdot 2\zeta\omega_n\omega}{(\omega_n^2-\omega^2)^2 + 4\zeta^2\omega_n^2\omega^2} \tag{51}$$

zatem $Q(\omega) < 0$, zaś $P(\omega) > 0$ dla $\omega < \omega_n$ lub $P(\omega) < 0$ dla $\omega > \omega_n$

$$G(j\omega) = \left|G(j\omega)\right| e^{j\varphi(\omega)} \tag{52}$$

$$\left|G(j\omega)\right| = \frac{k\omega_n^2}{\sqrt{(\omega_n^2-\omega^2)^2+4\zeta^2\omega_n^2\omega^2}}, \qquad \varphi(\omega) = -\mathrm{arctg}\frac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}$$



Rys. 47.

$$K(\omega) = k^2\omega_n^4/\left|G(j\omega)\right|^2 = (\omega_n^2-\omega^2)^2+4\zeta^2\omega_n^2\omega^2$$

$$\begin{aligned}\frac{dK(\omega)}{d\omega} &= 2(\omega_n^2-\omega^2)(-2\omega)+8\zeta^2\omega_n^2\omega = \\ &= -4\omega_n^2\omega+4\omega^3+8\zeta^2\omega_n^2\omega = 4\omega\left(\omega^2+2\zeta^2\omega_n^2-\omega_n^2\right)\end{aligned}$$

$$\omega = 0 \quad \vee \quad \omega^2 = \omega_n^2(1-2\zeta^2) \rightarrow \omega_r = \omega_n\sqrt{1-2\zeta^2}$$

przy puls. rezonansowej ω_r moduł $|G(j\omega)|$ osiąga maksimum:

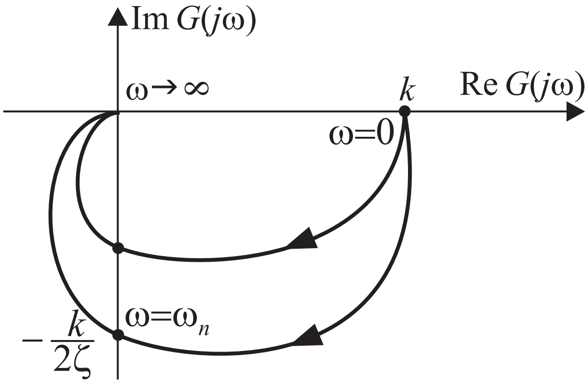
$$M_r = \max_{\omega}\left|G(j\omega)\right| = \left|G(j\omega_r)\right| = \frac{k\omega_n^2}{\sqrt{[\omega_n^2-\omega_n^2(1-2\zeta^2)]^2+4\zeta^2\omega_n^4(1-2\zeta^2)}} =$$

$$= \frac{k\omega_n^2}{\sqrt{4\omega_n^4\zeta^4+4\omega_n^4\zeta^2-8\omega_n^4\zeta^4}} = \frac{k\omega_n^2}{\sqrt{4\omega_n^4\zeta^2-4\omega_n^4\zeta^4}} = \frac{k}{2\zeta\sqrt{1-\zeta^2}} \tag{53}$$

$$\varphi(\omega_r) = -\mathrm{arctg}\frac{2\zeta\omega_n\omega_r}{\omega_n^2-\omega_r^2} = -\mathrm{arctg}\frac{2\zeta\omega_n^2\sqrt{1-2\zeta^2}}{\omega_n^2(1-1+2\zeta^2)} = -\mathrm{arctg}\frac{\sqrt{1-2\zeta^2}}{\zeta} \tag{54}$$

$\omega_r = \omega_n\sqrt{1-2\zeta^2}$ $\rightarrow$ rezonans występuje jedynie dla $0 \leqslant \zeta < \dfrac{\sqrt{2}}{2}$

$$Q(\omega_n) = \mathrm{Im}\{G(\omega_n)\} = -\frac{k}{2\zeta} \Rightarrow \begin{cases} |Q(\omega_n)| \leqslant \frac{1}{2}P(0) & \text{inerc. 2-go rz.} \\ |Q(\omega_n)| > \frac{1}{2}P(0) & \text{oscylacyjny} \end{cases}$$



Rys. 48.

17. Charakterystyki logarytmiczne elementu oscylacyjnego

$$\mathrm{Lm}(\omega) = 20\lg|G(j\omega)| = 20\lg(k\omega_n^2) - 20\lg\sqrt{(\omega_n^2-\omega^2)^2 + 4\zeta^2\omega_n^2\omega^2} =$$

$$= \begin{cases} 20\lg k + 20\lg\omega_n^2 - 20\lg\omega_n^2 & \text{dla } \omega \ll \omega_n \\ 20\lg k + 20\lg\omega_n^2 - 20\lg\omega^2 & \text{dla } \omega \gg \omega_n \end{cases} =$$

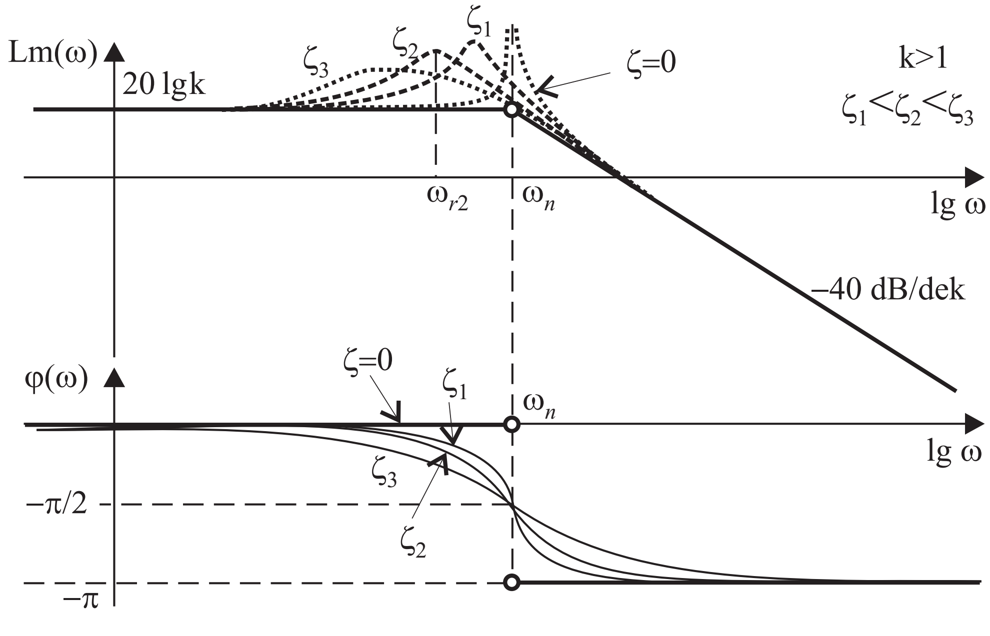
$$= \begin{cases} 20\lg k & \text{dla } \omega \ll \omega_n \\ 20\lg k - 40\lg\dfrac{\omega}{\omega_n} & \text{dla } \omega \gg \omega_n \end{cases} \qquad (k>0) \qquad (55)$$

nachylenie $\mathrm{Lm}(\omega)$ dla $\omega \gg \omega_n$ wynosi $-40\left[\frac{\mathrm{dB}}{\mathrm{dek}}\right]$

$$\mathrm{Lm}(\omega_r) = 20\lg k - 20\lg\left(2\zeta\sqrt{1-\zeta^2}\right) \qquad (56)$$

czyli $\zeta \searrow \quad \Rightarrow \quad \mathrm{Lm}(\omega_r) \nearrow$

$$\varphi(\omega) = -\mathrm{arctg}\frac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}$$



Rys. 49.