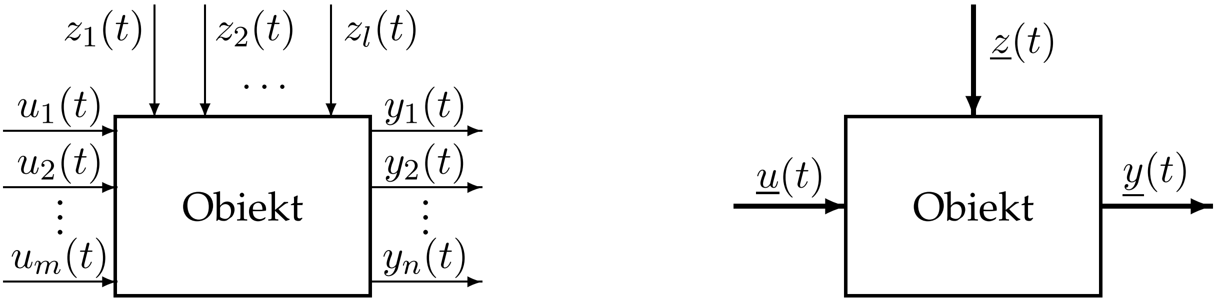
1. Podstawowe pojęcia

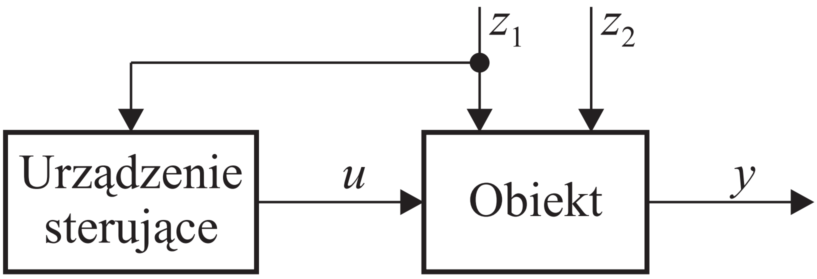


(a) Rys. 1 (b)

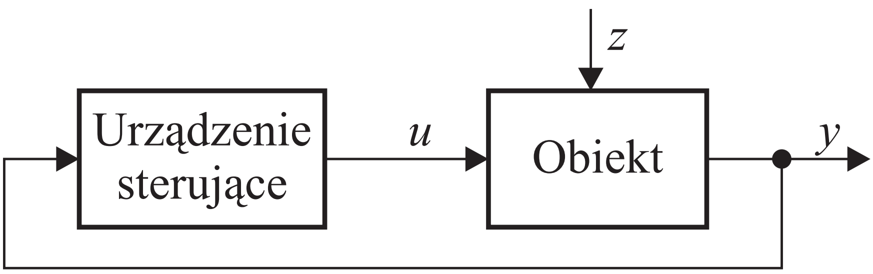
$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_m \end{bmatrix}, \qquad \underline{z} = \begin{bmatrix} z_1 \\ z_2 \\ \dots \\ z_l \end{bmatrix}, \qquad \underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix} \tag{1}$$

na ogół $m \neq l \neq n$; gdy $m = l = n = 1$, to układ jednowymiarowy
gdy $m \neq 1$ lub $l \neq 1$ lub $n \neq 1$ – układ wielowymiarowy

2. Otwarty i zamknięty układ sterowania



Rys. 2 Układ otwarty



Rys. 3 Układ zamknięty

3. Opis układu typu wejście-wyjście

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = \tag{2}$$
$$= b_m \frac{d^m u(t)}{dt^m} + b_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}} + \cdots + b_1 \frac{du(t)}{dt} + b_0 u(t)$$

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 \dot{y}(t) + a_0 y(t) = \tag{3}$$
$$= b_m u^{(m)}(t) + b_{m-1} u^{(m-1)}(t) + \cdots + b_1 \dot{u}(t) + b_0 u(t)$$

$m \leqslant n$; warunki początkowe:

$$y^{(i)}(0), \quad i = 0, 1, \dots n-1$$
$$u^{(j)}(0), \quad j = 0, 1, \dots m-1$$

Rys. 4

4. Transmitancja operatorowa

$$U(s) = \mathcal{L}[u(t)] = \int\limits_0^\infty u(t) e^{-st} dt, \quad Y(s) = \mathcal{L}[y(t)] = \int\limits_0^\infty y(t) e^{-st} dt$$

Przypomnijmy, że:

$$\boxed{\mathcal{L}[f^{(n)}(t)] = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0^+)}$$

przy czym $f^{(k)}(0^+) = \lim\limits_{t \to 0^+} \frac{df^{(k)}(t)}{dt^k}$

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 \dot{y}(t) + a_0 y(t) =$$
$$= b_m u^{(m)}(t) + b_{m-1} u^{(m-1)}(t) + \cdots + b_1 \dot{u}(t) + b_0 u(t)$$

$$a_n s^n Y(s) + a_{n-1} s^{n-1} Y(s) + \cdots + a_1 s Y(s) + a_0 Y(s) = \tag{4}$$
$$= b_m s^m U(s) + b_{m-1} s^{m-1} U(s) + \cdots + b_1 s U(s) + b_0 U(s)$$

$$Y(s)(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0) =$$
$$= U(s)(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0)$$

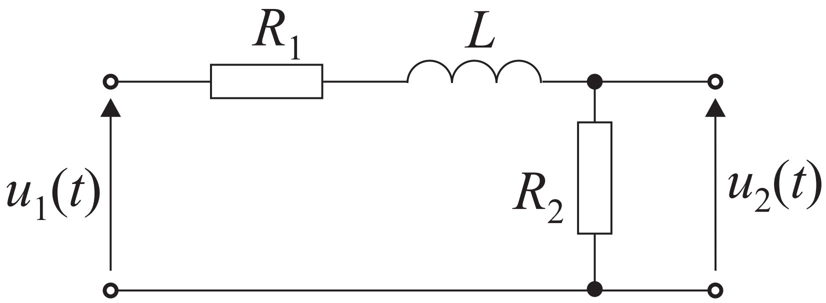
$$Y(s) = \frac{\sum\limits_{j=0}^{m} b_j s^j}{\sum\limits_{i=0}^{n} a_i s^i} U(s)$$

$$G(s) \triangleq \frac{Y(s)}{U(s)} = \frac{\sum\limits_{j=0}^{m} b_j s^j}{\sum\limits_{i=0}^{n} a_i s^i} \tag{5}$$

przy czym

$y^{(i)}(0+) = 0$, $i = 0, 1, \ldots n-1$ $\quad u^{(j)}(0+) = 0$, $j = 0, 1, \ldots m-1$

Przykład



Rys. 5

$$\begin{cases} u_1(t) = R_1 i(t) + L\dfrac{di(t)}{dt} + R_2 i(t) \\ u_2(t) = R_2 i(t) \end{cases}$$

Eliminujemy $i(t)$:

$$\frac{L}{R_2}\frac{du_2(t)}{dt} + \frac{R_1}{R_2}u_2(t) + u_2(t) = u_1(t)$$

$$\frac{L}{R_2}\dot{u}_2(t) + \left(1 + \frac{R_1}{R_2}\right)u_2(t) = u_1(t)$$

$$\left(s\frac{L}{R_2} + 1 + \frac{R_1}{R_2}\right)U_2(s) = U_1(s)$$

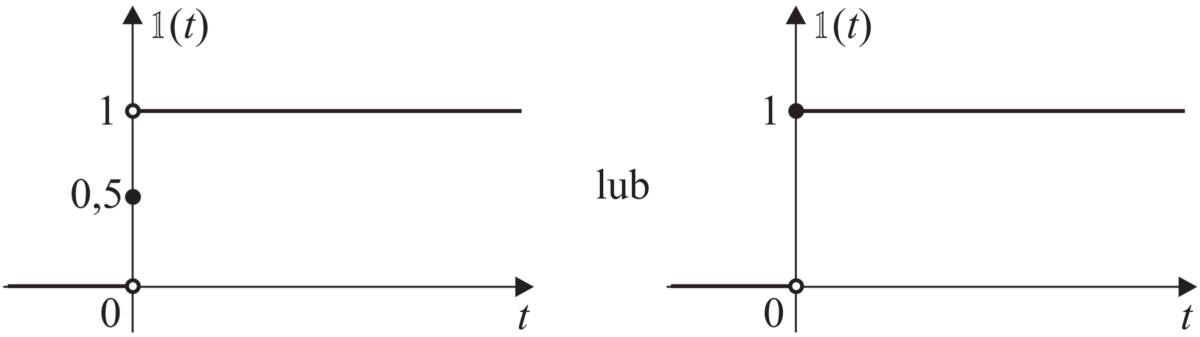
$$G(s) = \frac{U_2(s)}{U_1(s)} = \frac{1}{s\frac{L}{R_2} + 1 + \frac{R_1}{R_2}} = \frac{R_2}{sL + R_1 + R_2} = \frac{\frac{R_2}{R_1+R_2}}{s\frac{L}{R_1+R_2} + 1}$$

$$k = \frac{R_2}{R_1 + R_2}, \quad T = \frac{L}{R_1 + R_2} \quad \rightarrow \quad G(s) = \frac{k}{1 + sT}$$

(element inercyjny 1-go rzędu)

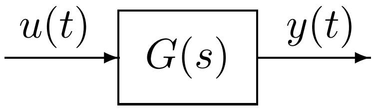
5. Odpowiedź skokowa

$$\mathbb{1}(t) = \begin{cases} 1 & \text{dla } t > 0 \\ \frac{1}{2} & \text{dla } t = 0 \\ 0 & \text{dla } t < 0 \end{cases} \quad \text{lub uproszcz.} \quad \mathbb{1}(t) = \begin{cases} 1 & \text{dla } t \geqslant 0 \\ 0 & \text{dla } t < 0 \end{cases} \tag{6}$$



Rys. 6

$$\mathcal{L}[\mathbb{1}(t)] = \frac{1}{s}$$



Rys. 7

$$U(s) = \mathcal{L}[u(t)], \qquad Y(s) = \mathcal{L}[y(t)]$$

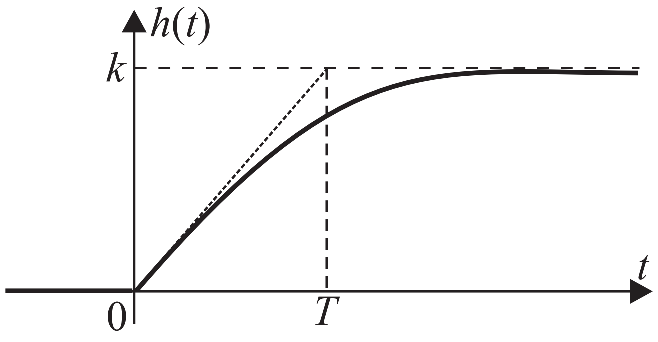
$$Y(s) = G(s)U(s) \qquad \text{ponieważ} \qquad G(s) = \frac{Y(s)}{U(s)}$$

$$u(t) = \mathbb{1}(t) \quad \rightarrow \quad H(s) = G(s)\frac{1}{s} \quad \rightarrow \quad h(t) = \mathcal{L}^{-1}\left[\frac{G(s)}{s}\right]$$

Przykład $G(s) = \dfrac{k}{1+sT}$

$$h(t) = \mathcal{L}^{-1}\left[\frac{k}{1+sT}\frac{1}{s}\right] = \mathcal{L}^{-1}\left[\frac{k}{T}\frac{1}{s(s+\frac{1}{T})}\right] = \frac{k}{T}\left(\lim_{s\to 0}\frac{1}{s+\frac{1}{T}}e^{st} + \right.$$

$$\left. + \lim_{s\to -1/T}\frac{1}{s}e^{st}\right) = \frac{k}{T}\left(T - Te^{-\frac{t}{T}}\right)\mathbb{1}(t) = k\left(1 - e^{-\frac{t}{T}}\right)\mathbb{1}(t)$$



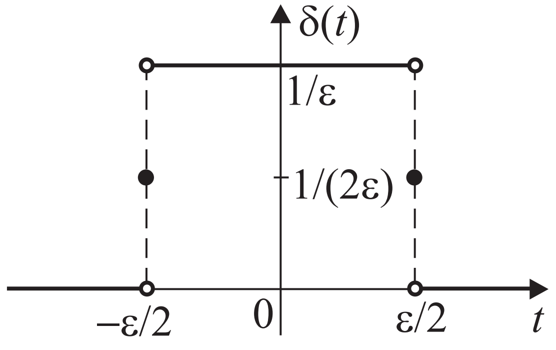
Rys. 8

6. Odpowiedź impulsowa

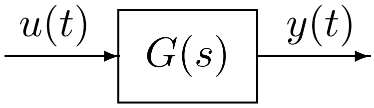
$$\delta(t) = \begin{cases} 0 & \text{dla } t \neq 0 \\ \infty & \text{dla } t = 0 \end{cases} \qquad \int\limits_{-\infty}^{\infty} \delta(t)dt = 1 \tag{7}$$

lub inaczej:

$$\delta(t) = \begin{cases} 1/\varepsilon & \text{dla } t \in (-\varepsilon/2, \varepsilon/2) \\ 1/(2\varepsilon) & \text{dla } t = \pm\varepsilon/2 \\ 0 & \text{dla } |t| > \varepsilon/2 \end{cases} \tag{8}$$



Rys. 9



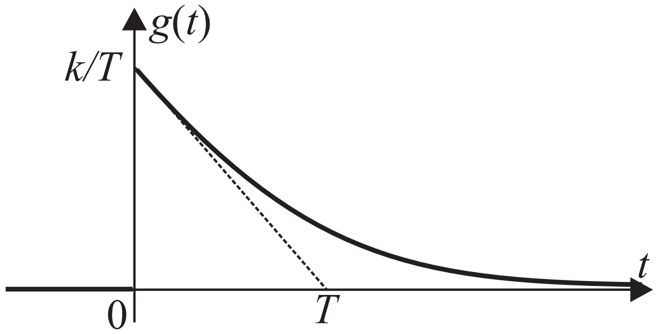
Rys. 10

$$Y(s) = G(s)U(s) = G(s) \cdot 1 \qquad \mathcal{L}[\delta(t)] = 1$$

$$g(t) = y(t) = \mathcal{L}^{-1}[G(s)]$$

Przykład $\qquad G(s) = \dfrac{k}{1 + sT}$

$$g(t) = \mathcal{L}^{-1}\left[\frac{k}{1+sT}\right] = \mathcal{L}^{-1}\left[\frac{k}{T}\frac{1}{s+\frac{1}{T}}\right] = \frac{k}{T}e^{-\frac{t}{T}}\mathbb{1}(t)$$



Rys. 11

7. Związek między $h(t)$ i $g(t)$

$$h(t) = \int_0^t g(\tau)d\tau, \qquad g(t) = \frac{d}{dt}h(t) \tag{9}$$

8. Całka splotowa

$$Y(s) = G(s)U(s)$$

$$y(t) = \int_0^t u(\tau)g(t-\tau)d\tau = \int_0^t u(t-\tau)g(\tau)d\tau \tag{10}$$

Przykład (element całkujący idealny i rzeczywisty)

$$G_1(s) = \frac{Y_1(s)}{U(s)} = \frac{k}{s}, \qquad G_2(s) = \frac{Y_2(s)}{U(s)} = \frac{k}{s(1+sT)}$$

równania różniczkowe

$$kU(s) = sY_1(s) \qquad kU(s) = sY_2(s) + Ts^2Y_2(s)$$

$$ku(t) = \dot{y}_1(t) \qquad ku(t) = T\ddot{y}_2(t) + \dot{y}_2(t)$$

odpowiedzi skokowe

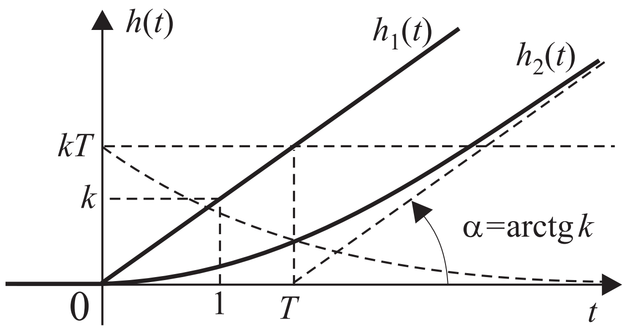
$$h_1(t) = \mathcal{L}^{-1}\left[G_1(s)\frac{1}{s}\right] = \mathcal{L}^{-1}\left[\frac{k}{s^2}\right] = kt\mathbb{1}(t)$$

$$h_2(t) = \mathcal{L}^{-1}\left[G_2(s)\frac{1}{s}\right] = \mathcal{L}^{-1}\left[\frac{k}{s^2(1+sT)}\right] = \frac{k}{T}\mathcal{L}^{-1}\left[\frac{1}{s^2(s+\frac{1}{T})}\right] =$$

$$= \frac{k}{T}\left[\lim_{s\to 0}\frac{d}{ds}\left(\frac{1}{s+\frac{1}{T}}e^{st}\right) + \lim_{s\to -\frac{1}{T}}\frac{1}{s^2}e^{st}\right]\mathbb{1}(t) =$$

$$= \frac{k}{T}\left[\lim_{s\to 0}\frac{te^{st}(s+\frac{1}{T}) - e^{st}}{(s+\frac{1}{T})^2} + T^2e^{-\frac{t}{T}}\right]\mathbb{1}(t) =$$

$$= \frac{k}{T}\left[T^2\left(\frac{t}{T}-1\right) + T^2e^{-\frac{t}{T}}\right]\mathbb{1}(t) = k(t - T + Te^{-\frac{t}{T}})\mathbb{1}(t)$$

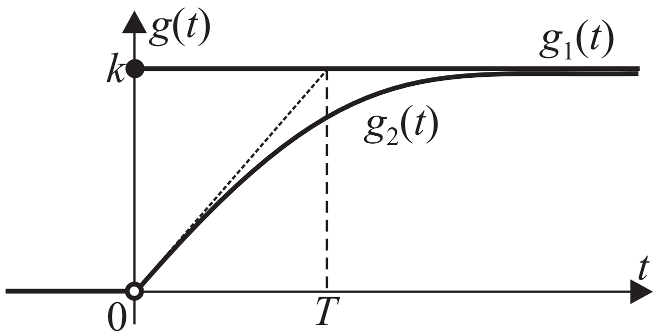


Rys. 12

odpowiedzi impulsowe:

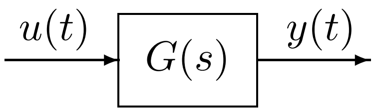
$$g_1(t) = \mathcal{L}^{-1}\left[G_1(s)\right] = \mathcal{L}^{-1}\left[\frac{k}{s}\right] = k\mathbb{1}(t)$$

$$g_2(t) = \mathcal{L}^{-1}\left[G_2(s)\right] = \mathcal{L}^{-1}\left[\frac{k}{s(1+sT)}\right] = k(1-e^{-\frac{t}{T}})\mathbb{1}(t)$$

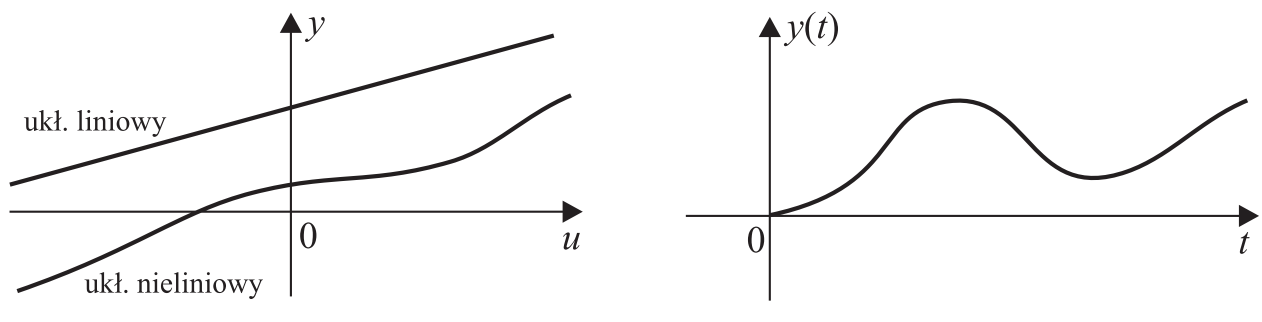


Rys. 13

9. Charakterystyki statyczne i dynamiczne



Rys. 14

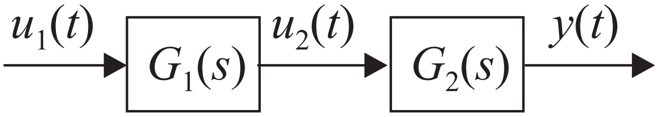


(a) ch-ki statyczne (b) ch-ka dynamiczna

Rys. 15

10. Wyznaczanie transmitancji wypadkowych

a) połączenie szeregowe (kaskadowe)

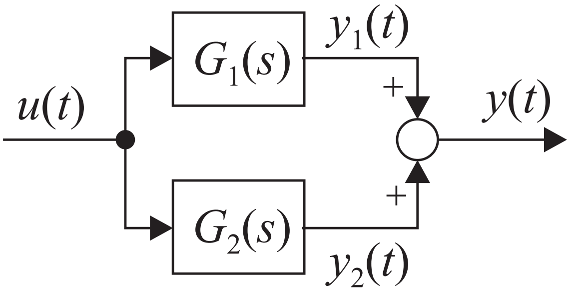


Rys. 16

$$G_1(s) = \frac{U_2(s)}{U_1(s)}, \qquad G_2(s) = \frac{Y(s)}{U_2(s)}$$

$$G(s) = \frac{Y(s)}{U_1(s)} = \frac{Y(s)}{U_2(s)}\frac{U_2(s)}{U_1(s)} = G_1(s)G_2(s) \tag{11}$$

b) połączenie równoległe

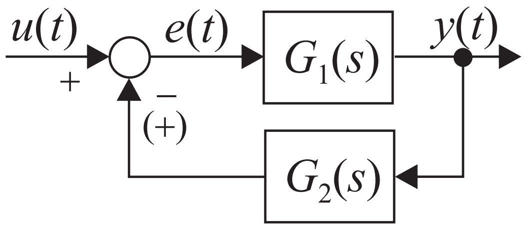


Rys. 17

$$\begin{aligned} Y(s) &= Y_1(s) + Y_2(s) = G_1(s)U(s) + G_2(s)U(s) = \\ &= (G_1(s) + G_2(s))U(s) \end{aligned}$$

$$G(s) = \frac{Y(s)}{U(s)} = G_1(s) + G_2(s) \tag{12}$$

c) sprzężenie zwrotne



Rys. 18

$$Y(s) = G_1(s)E(s) = G_1(s)\left[U(s) - G_2(s)Y(s)\right]$$

$$Y(s) + G_1(s)G_2(s)Y(s) = G_1(s)U(s)$$

$$Y(s)\left[1 + G_1(s)G_2(s)\right] = G_1(s)U(s)$$

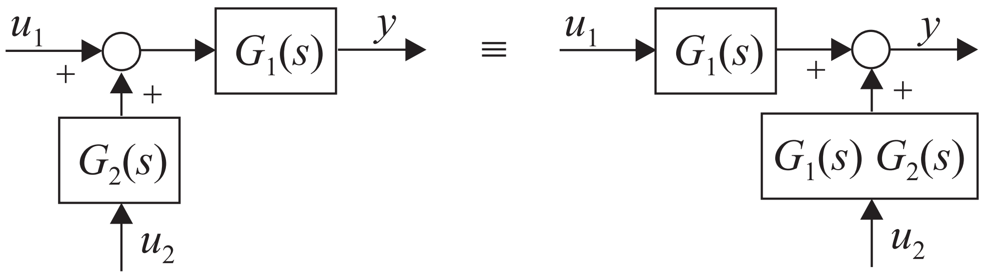
$$G(s) = \frac{Y(s)}{U(s)} = \frac{G_1(s)}{1 + G_1(s)G_2(s)} \tag{13}$$

dla dodatniego sprzężenia zwrotnego: $G(s) = \frac{G_1(s)}{1 - G_1(s)G_2(s)}$
gdy $G_2(s) = 1$ – bezpośrednie sprzężenie zwrotne i wtedy: $G(s) = \frac{G_1(s)}{1 \pm G_1(s)}$

11. Przekształcanie schematów blokowych

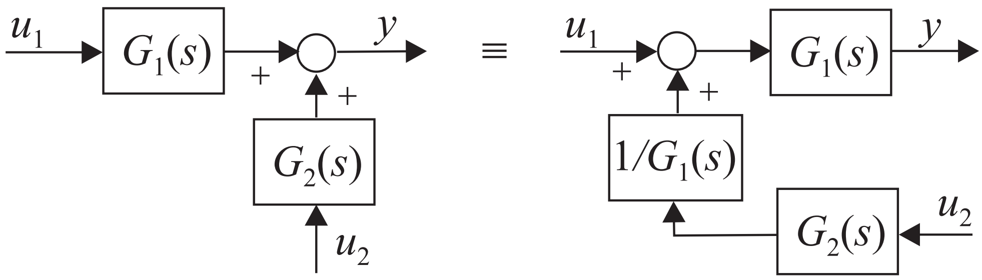
a) przenoszenie węzła sumacyjnego z we na wy i odwrotnie

$$G_1(s)\left[U_1(s) + G_2(s)U_2(s)\right] = G_1(s)U_1(s) + G_1(s)G_2(s)U_2(s)$$



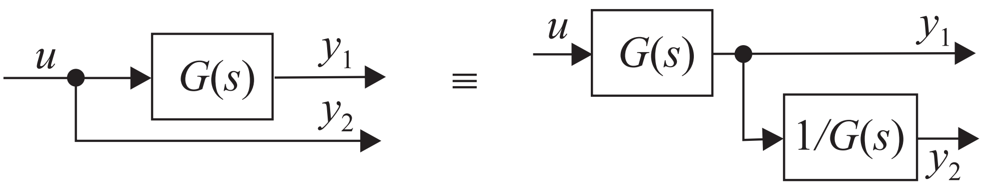
Rys. 19

$$G_1(s)U_1(s) + G_2(s)U_2(s) = G_1(s)\left[U_1(s) + \frac{1}{G_1(s)}G_2(s)U_2(s)\right]$$

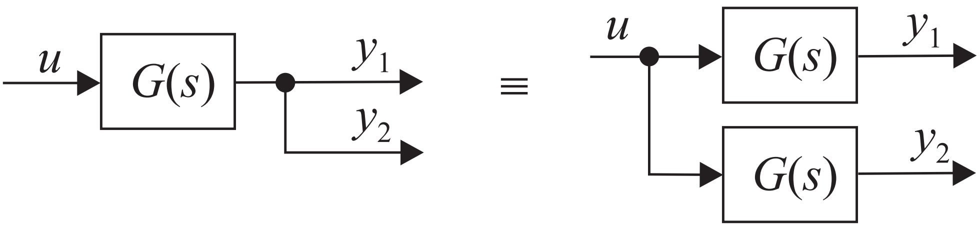


Rys. 20

b) przenoszenie węzła informacyjnego (rozgałęźnego) z we na wy i odwrotnie



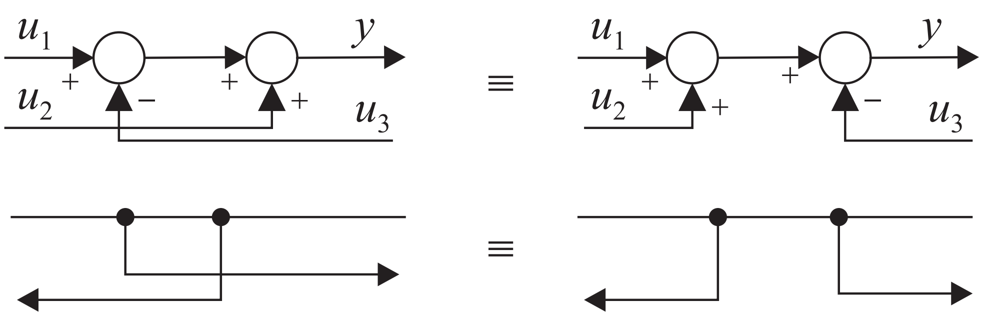
Rys. 21



Rys. 22

c) zamiana miejsc węzłów sąsiadujących ze sobą

$$y = (u_1 - u_3) + u_2 = (u_1 + u_2) - u_3$$



Rys. 23

12. Transmitancja widmowa

$$G(j\omega) = G(s)|_{s=j\omega}, \qquad \omega = 2\pi f \tag{14}$$

$$\xrightarrow{u(t)} \boxed{G(j\omega)} \xrightarrow{y(t)}$$

Rys. 24

$$y(t) = y_p(t) + y_u(t), \qquad \lim_{t\to\infty} y_p(t) = 0 \tag{15}$$

$$u(t) = A \sin \omega t \cdot \mathbb{1}(t) \Rightarrow$$

$$\Rightarrow y(t) = y_u(t) = A|G(j\omega)| \sin(\omega t + \varphi(\omega)) \cdot \mathbb{1}(t) \tag{16}$$

Przykład

$$G(s) = \frac{2}{s^2 + 3s + 2} = \frac{2}{(s+1)(s+2)} = \frac{1}{(s+1)(0{,}5s+1)}$$

$u(t) = 8 \sin 2t \cdot \mathbb{1}(t) \rightarrow$ wyznaczyć przebieg $y_u(t)$

$$G(s)|_{s=j2} = \frac{2}{(j2)^2 + 3(j2) + 2} = \frac{2}{-2 + j6} =$$

$$= \frac{2}{\sqrt{(-2)^2 + 6^2}} e^{-j \operatorname{arctg} \frac{6}{-2}} = 0{,}316\, e^{-j108{,}4^\circ}$$

$$y_u(t) = 8 \cdot 0{,}316 \sin(2t - 108{,}4^\circ) \cdot \mathbb{1}(t) =$$

$$= 2{,}528 \sin(2t - 108{,}4^\circ) \cdot \mathbb{1}(t)$$

13. Charakterystyki częstotliwościowe

$\omega \in \langle 0, +\infty)$

- ch. amplitudowo-fazowa (wykres Nyquista),
- ch-ki logarytmiczne (wykresy Bodego).

14. Charakterystyka amplitudowo-fazowa

$$G(s)=\frac{b_m s^m + b_{m-1}s^{m-1}+\cdots+b_1 s+b_0}{a_n s^n + a_{n-1}s^{n-1}+\cdots+a_1 s+a_0},\quad m\leqslant n \tag{17}$$

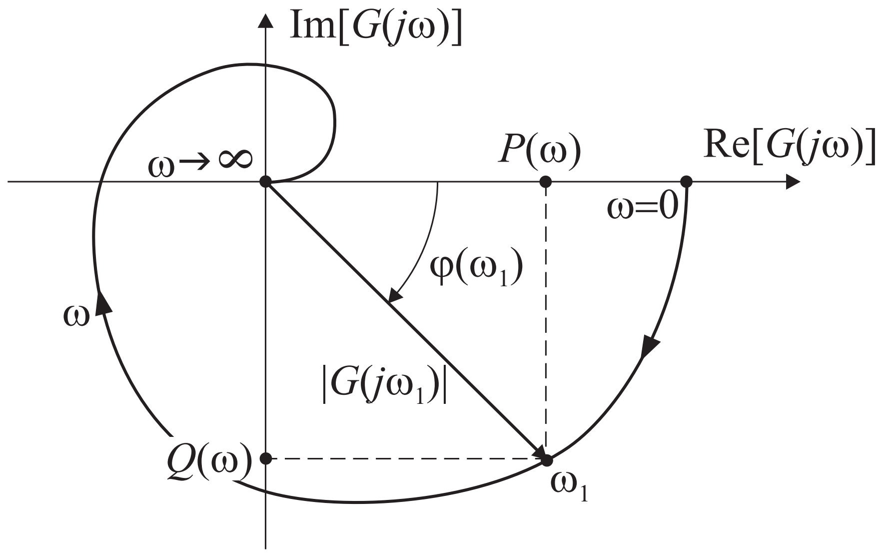
$$G(j\omega) = G(s)|_{s=j\omega} = \tag{18}$$

$$= \frac{b_m(j\omega)^m + b_{m-1}(j\omega)^{m-1}+\cdots+b_1(j\omega)+b_0}{a_n(j\omega)^n + a_{n-1}(j\omega)^{n-1}+\cdots+a_1(j\omega)+a_0}$$

$$G(j\omega) = P(\omega) + jQ(\omega) = |G(j\omega)|e^{j\varphi(\omega)} \tag{19}$$

$$P(\omega) = \mathrm{Re}[G(j\omega)], \qquad |G(j\omega)| = \sqrt{P^2(\omega)+Q^2(\omega)},$$

$$Q(\omega) = \mathrm{Im}[G(j\omega)], \qquad \varphi(\omega) = \mathrm{arctg}\frac{Q(\omega)}{P(\omega)}.$$



Rys. 25 $(0 \leqslant \omega < \infty)$

$$u(t) = A\sin\omega_1 t \Rightarrow y(t) = A|G(j\omega_1)|\sin(\omega_1 t + \varphi(\omega_1))$$

Przykład (element inercyjny 1-go rzędu)

$$G(j\omega) = \frac{k}{1+j\omega T} = \frac{k(1-j\omega T)}{1+\omega^2T^2} =$$

$$= \frac{k}{1+\omega^2T^2} - j\frac{k\omega T}{1+\omega^2T^2} = P(\omega)+jQ(\omega)$$

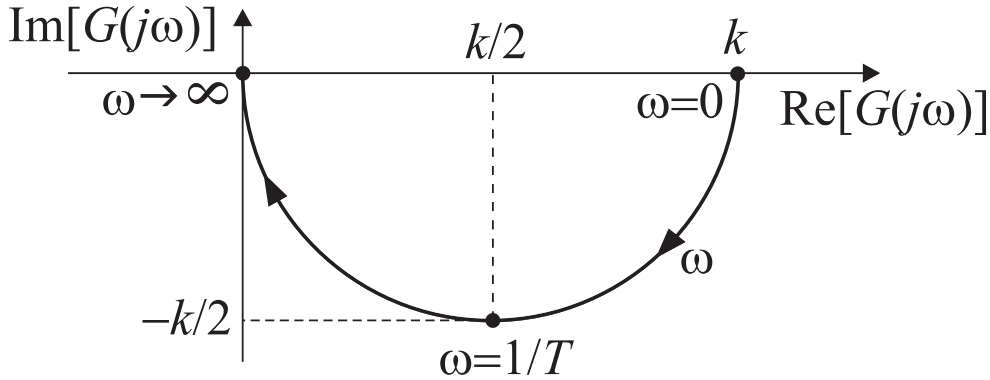
$$\omega > 0 \quad \rightarrow \quad P>0,\ Q\leqslant 0 \quad \rightarrow \quad \text{IV ćw.}$$

$$P = \frac{k}{1+\omega^2T^2} \;\rightarrow\; 1+\omega^2T^2 = \frac{k}{P} \;\rightarrow\; \omega^2T^2 = \frac{k-P}{P}$$

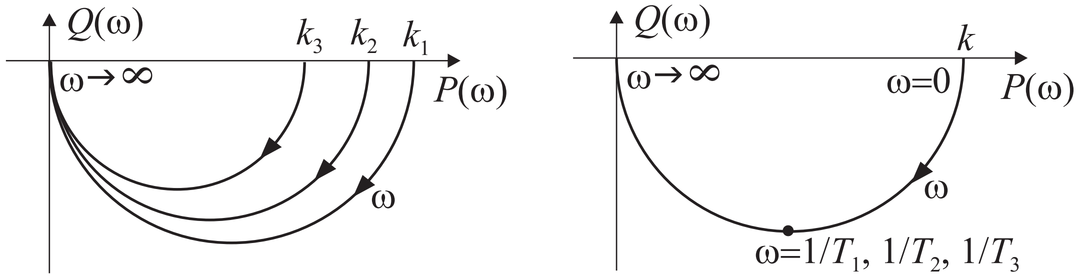
$$Q^2 = \frac{k^2\omega^2T^2}{(1+\omega^2T^2)^2} = \frac{k^2\dfrac{k-P}{P}}{k^2/P^2} = P(k-P)$$

$$Q^2 + P^2 - kP = 0 \;\rightarrow\; P^2 - kP + \frac{k^2}{4} + Q^2 = \frac{k^2}{4}$$

$$\left(P - \frac{k}{2}\right)^2 + Q^2 = \left(\frac{k}{2}\right)^2, \qquad Q \leqslant 0$$



Rys. 26



(a) $k = \text{var}$, $k_3 < k_2 < k_1$ (b) $T = \text{var}$

Rys. 27